MIT 6.02 **DRAFT** *Lecture Notes*
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CHAPTER 13
Fourier Analysis and Spectral Representation of Signals

We have seen in the previous chapter that the action of an LTI system on a sinusoidal or complex exponential input signal can be represented effectively by the frequency response $H(\Omega)$ of the system. By superposition, it then becomes easy—again using the frequency response—to determine the action of an LTI system on a *weighted linear combination of sinusoids or complex exponentials* (as illustrated in Example 3 of the preceding chapter). The natural question now is how large a class of signals can be represented in this manner. The short answer to this question: most signals you are likely to be interested in!

The tool for exposing the decomposition of a signal into a weighted sum of sinusoids or complex exponentials is **Fourier analysis**. We first discuss the Discrete-Time Fourier Transform (DTFT), which we have actually seen hints of already and which applies to the most general classes of signals. We then move to the Discrete-Time Fourier Series (DTFS), which constructs a similar representation for the special case of periodic signals, or for signals of finite duration. The DTFT development provides some useful background, context and intuition for the more special DTFS development, but may be skimmed over on an initial reading (i.e., understand the logical flow of the development, but don't struggle too much with the mathematical details).

■ 13.1 The Discrete-Time Fourier Transform

We have in fact already derived an expression in the previous chapter that has the flavor of what we are looking for. Recall that we obtained the following representation for the unit sample response $h[n]$ of an LTI system:

$$h[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) e^{j\Omega n} \, d\Omega \; , \tag{13.1}$$

where the frequency response, $H(\Omega)$, was defined by

$$H(\Omega) = \sum_{m=-\infty}^{\infty} h[m]e^{-j\Omega m} \, . \tag{13.2}$$

Equation (13.1) can be interpreted as representing the signal $h[n]$ by a weighted combination of a continuum of exponentials, of the form $e^{j\Omega n}$, with frequencies Ω in a 2π-range, and associated weights $H(\Omega)\, d\Omega$.

As far as these expressions are concerned, the signal $h[n]$ is fairly arbitrary; the fact that we were considering it as the unit sample response of a system was quite incidental. We only required it to be a signal for which the infinite sum on the right of Equation (13.2) was well-defined. We shall accordingly rewrite the preceding equations in a more neutral notation, using $x[n]$ instead of $h[n]$:

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega)e^{j\Omega n}\, d\Omega \, , \tag{13.3}$$

where $X(\Omega)$ is defined by

$$X(\Omega) = \sum_{m=-\infty}^{\infty} x[m]e^{-j\Omega m} \, . \tag{13.4}$$

For a general signal $x[\cdot]$, we refer to the 2π-periodic quantity $X(\Omega)$ as the **discrete-time Fourier transform (DTFT)** of $x[\cdot]$; it would no longer make sense to call it a frequency response. Even when the signal is real, the DTFT will in general be complex at each Ω.

The DTFT *synthesis* equation, Equation (13.3), shows how to synthesize $x[n]$ as a weighted combination of a continuum of exponentials, of the form $e^{j\Omega n}$, with frequencies Ω in a 2π-range, and associated weights $X(\Omega)\, d\Omega$. From now on, unless mentioned otherwise, we shall take Ω to lie in the range $[-\pi, \pi]$.

The DTFT *analysis* equation, Equation (13.4), shows how the weights are determined. We also refer to $X(\Omega)$ as the *spectrum* or *spectral distribution* or *spectral content* of $x[\cdot]$.

Example 1 (Spectrum of Unit Sample Function) Consider the signal $x[n] = \delta[n]$, the unit sample function. From the definition in Equation (13.4), the spectral distribution is given by $X(\Omega) = 1$, because $x[n] = 0$ for all $n \neq 0$, and $x[0] = 1$. The spectral distribution is thus constant at the value 1 in the entire frequency range $[-\pi, \pi]$. What this means is that it takes the addition of *equal* amounts of complex exponentials at *all* frequencies in a 2π-range to synthesize a unit sample function, a perhaps surprising result. What's happening here is that all the complex exponentials reinforce each other at time $n = 0$, but effectively cancel each other out at every other time instant.

Example 2 (Phase Matters) What if $X(\Omega)$ has the same magnitude as in the previous example, so $|X(\Omega)| = 1$, but has a nonzero phase characteristic, $\angle X(\Omega) = -\alpha\Omega$ for some $\alpha \neq 0$? This phase characteristic is *linear* in Ω. With this,

$$X(\Omega) = 1.e^{-j\alpha\Omega} = e^{-j\alpha\Omega} \, .$$

To find the corresponding time signal, we simply carry out the integration in Equation (13.3). If α is an *integer*, the integral

$$x[n] = \frac{1}{2\pi}\int_{<2\pi>} e^{-j\alpha\Omega}e^{j\Omega n}\,d\Omega = \frac{1}{2\pi}\int_{<2\pi>} e^{j(n-\alpha)\Omega}\,d\Omega$$

yields the value 0 for all $n \neq \alpha$. To see this, note that

$$e^{j(n-\alpha)\Omega} = \cos\Big((n-\alpha)\Omega\Big) + j\sin\Big((n-\alpha)\Omega\Big)\,,$$

and the integral of this expression over any 2π-interval is 0, when $n-\alpha$ is a nonzero integer. However, if $n-\alpha = 0$, i.e., if $n = \alpha$, the cosine evaluates to 1, the sine evaluates to 0, and the integral above evaluates to 1. We therefore conclude that when α is an integer,

$$x[n] = \delta[n-\alpha]\,.$$

The signal is just a shifted unit sample (delayed by α if $\alpha > 0$, and advanced by $|\alpha|$ otherwise). The effect of adding the phase characteristic to the case in Example 1 has been to just shift the unit sample in time.

For *non-integer* α, the answer is a little more intricate:

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\alpha\Omega}e^{j\Omega n}\,d\Omega \\ &= \frac{1}{2\pi}\frac{e^{j(n-\alpha)\Omega}}{j(n-\alpha)}\Big|_{-\pi}^{\pi} \\ &= \frac{\sin\Big(\pi(n-\alpha)\Big)}{\pi(n-\alpha)} \end{aligned}$$

This time-function is referred to as a "sinc" function. We encountered this function when determining the unit sample response of an ideal lowpass filter in the previous chapter.

Example 3 (A Bandlimited Signal) Consider now a signal whose spectrum is flat but band-limited:

$$X(\Omega) = \begin{cases} 1 & \text{for} \quad |\Omega| < \Omega_c \\ 0 & \text{for} \quad \Omega_c \leq |\Omega| \leq \pi \end{cases}$$

The corresponding signal is again found directly from Equation (13.3). For $n \neq 0$, we get

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega n}\,d\Omega \\ &= \frac{1}{2\pi}\frac{e^{j\Omega}}{jn}\Big|_{-\Omega_c}^{\Omega_c} \\ &= \frac{\sin(\Omega_c n)}{\pi n}\,, \end{aligned} \qquad (13.5)$$

which is again a sinc function. For $n = 0$, Equation (13.3) yields

$$x[n] = \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} 1\, d\Omega = \frac{\Omega_c}{\pi} .$$

(This is exactly what we would get from Equation (13.5) if n was treated as a continuous variable, and the limit of the sinc function as $n \to 0$ was evaluated by L'Hôpital's rule—a useful mnemonic, but not a derivation!)

From our study of the analogous equations for $h[\cdot]$ in the previous chapter, we know that the DTFT of $x[\cdot]$ is well-defined when this signal is *absolutely summable*,

$$\sum_{m=-\infty}^{\infty} \Big| x[m] \Big| \leq \mu < \infty$$

for some μ. However, the DTFT is in fact well-defined for signals that satisfy less demanding constraints, for instance *square summable* signals,

$$\sum_{m=-\infty}^{\infty} \Big| x[m] \Big|^2 \leq \mu < \infty .$$

The sinc function in the examples above is actually *not* absolutely summable because it follows off too slowly—only as $1/n$—as $|n| \to \infty$. However, it *is* square summable.

A digression: One can also define the DTFT for signals $x[n]$ that do not converge to 0 as $|n| \to \infty$, provided they grow no faster than polynomially in n as $|n| \to \infty$. An example of such a signal of *slow growth* would be $x[n] = e^{j\Omega_0 n}$ for all n, whose spectrum must be concentrated at $\Omega = \Omega_0$. However, the corresponding $X(\Omega)$ turns out to no longer be an ordinary function, but is a (scaled) *Dirac impulse* in frequency, located at $\Omega = \Omega_0$:

$$X(\Omega) = 2\pi\delta(\Omega - \Omega_0) .$$

You may have encountered the Dirac impulse in other settings. The unit impulse at $\Omega = \Omega_0$ can be thought of as a "function" that has the value 0 at all points except at $\Omega = \Omega_0$, and has unit area. This is an instance of a broader result, namely that signals of slow growth possess transforms that are generalized functions (e.g., impulses), which have to be interpreted in terms of what they do under an integral sign, rather than as ordinary functions. It is partly in order to avoid having to deal with impulses and generalized functions in treating sinusoidal and periodic signals that we shall turn to the Discrete-Time Fourier *Series* rather than the DTFT. *End of digression!*

We make one final observation before moving to the DTFS. As shown in the previous chapter, if the input $x[n]$ to an LTI system with frequency response $H(\Omega)$ is the (everlasting) exponential signal $e^{j\Omega n}$, then the output is $y[n] = H(\Omega)e^{j\Omega n}$. By superposition, if the input is instead the weighted linear combination of such exponentials that is given in Equation (13.3), then the corresponding output must be the same weighted combination of *responses*, so

$$y[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega)X(\Omega)e^{j\Omega n}\, d\Omega . \tag{13.6}$$

However, we also know that the term $H(\Omega)X(\Omega)$ multiplying the complex exponential in this expression must be the DTFT of $y[\cdot]$, so

$$Y(\Omega) = H(\Omega)X(\Omega) \ . \tag{13.7}$$

Thus, the time-domain convolution relation $y[n] = (h * x)[n]$ has been converted to a simple multiplication in the frequency domain. This is a result we saw in the previous chapter too, when discussing the frequency response of a series or cascade combination of two LTI systems: the relation $h[n] = (h_1 * h_2)[n]$ in the time domain mapped to an overall frequency response of $H(\Omega) = H_1(\Omega)H_2(\Omega)$ that was simply the product of the individual frequency responses. This is a major reason for the power of frequency-domain analysis; the more involved operation of convolution in time is replaced by multiplication in frequency.

■ 13.2 The Discrete-Time Fourier Series

The DTFT synthesis expression in Equation (13.3) expressed $x[n]$ as a weighted sum of a *continuum* of complex exponentials, involving *all* frequencies Ω in $[-\pi, \pi]$. Suppose now that $x[n]$ is a *periodic* signal of (integer) period P, so

$$x[n+P] = x[n]$$

for all n. This signal is completely specified by the P values it takes in a single period, for instance the values $x[0], x[1], \ldots, x[P-1]$. It would seem in this case as though we should be able to get away with using a smaller number of complex exponentials to construct $x[n]$ on the interval $[0, P-1]$ and thereby for all n. The **discrete-time Fourier series (DTFS)** shows that this is indeed the case.

Before we write down the DTFS, a few words of reassurance are warranted. The expressions below may seem somewhat bewildering at first, with a profusion of symbols and subscripts, but once we get comfortable with what the expressions are saying, interpret them in different ways, and do some examples, they end up being quite straightforward. So don't worry if you don't get it all during the first pass through this material—allow yourself some time, and a few visits, to get comfortable!

■ 13.2.1 The Synthesis Equation

The essence of the DTFS is the following statement:

Any P-periodic signal $x[n]$ can be represented (or synthesized) as a weighted linear combination of P complex exponentials (or spectral components), where the frequencies of the exponentials are located evenly in the interval $[-\pi, \pi]$, starting in the middle at the frequency $\Omega_0 = 0$ and increasing outwards in both directions in steps of $\Omega_1 = 2\pi/P$.

More concretely, the claim is that any P-periodic DT signal $x[n]$ can be represented in the form

$$x[n] = \sum_{k=\langle P \rangle} A_k e^{j\Omega_k n} \ , \tag{13.8}$$

where we write $k = \langle P \rangle$ to indicate that k runs over *any* set of P consecutive integers. The

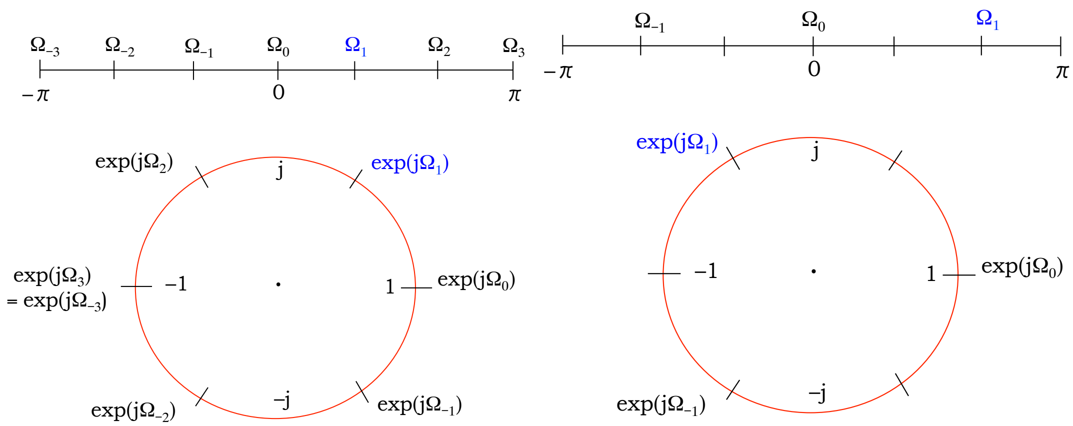

Figure 13-1: When P is even, the end frequencies are at $\pm\pi$ and the Ω_k values are as shown in the pictures on the left for $P = 6$. When P is odd, the end frequencies are at $\pm(\pi - \frac{\Omega_1}{2})$, as shown on the right for $P = 3$.

Fourier series coefficients or *spectral weights* A_k in this expression are complex numbers in general, and the *spectral frequencies* Ω_k are defined by

$$\Omega_k = k\Omega_1 \ , \quad \text{where} \quad \Omega_1 = \frac{2\pi}{P} \ . \tag{13.9}$$

We refer to Ω_1 as the *fundamental frequency* of the periodic signal, and to Ω_k as the k-th *harmonic*. Note that $\Omega_0 = 0$.

Note that the expression on the right side of Equation (13.8) does indeed repeat periodically every P time steps, because each of the constituent exponentials

$$e^{j\Omega_k n} = e^{jk\Omega_1 n} = e^{jkn(2\pi/P)} = \cos(k\frac{2\pi}{P}n) + j\sin(k\frac{2\pi}{P}n) \tag{13.10}$$

repeats when n changes by an integer multiple of P.

It also follows from Equation (13.10) that changing the *frequency* index k by P — or more generally by any positive or negative integer multiple of P — brings the exponential in that equation back to the same point on the unit circle, because the corresponding frequency Ω_k has then changed by an integer multiple of 2π. This is why it suffices to choose $k = \langle P \rangle$ in the DTFS representation.

Putting all this together, it follows that the frequencies of the complex exponentials used to synthesize a P-periodic signal $x[n]$ via the DTFS are located evenly in the interval $[-\pi, \pi]$, starting in the middle at the frequency $\Omega_0 = 0$ and increasing outwards in both directions in steps of $\Omega_1 = 2\pi/P$. In the case of an even value of P, such as the case $P = 6$ in Figure 13-1 (left), the end frequencies will be at $\pm\pi$ (we need only one of these frequencies, not both, as they translate to the same point on the unit circle when we write $e^{j\Omega_k n}$). In the case of an odd value of P, such as the case $P = 3$ shown in Figure 13-1 (right), the end points are $\pm(\pi - \frac{\Omega_1}{2})$.

The weights $\{A_k\}$ collectively constitute the **spectrum** of the periodic signal, and we typically plot them as a function of the frequency index k, as in Figure 13-2 The spectral weights in these simple sinusoidal examples have been determined by inspection, through

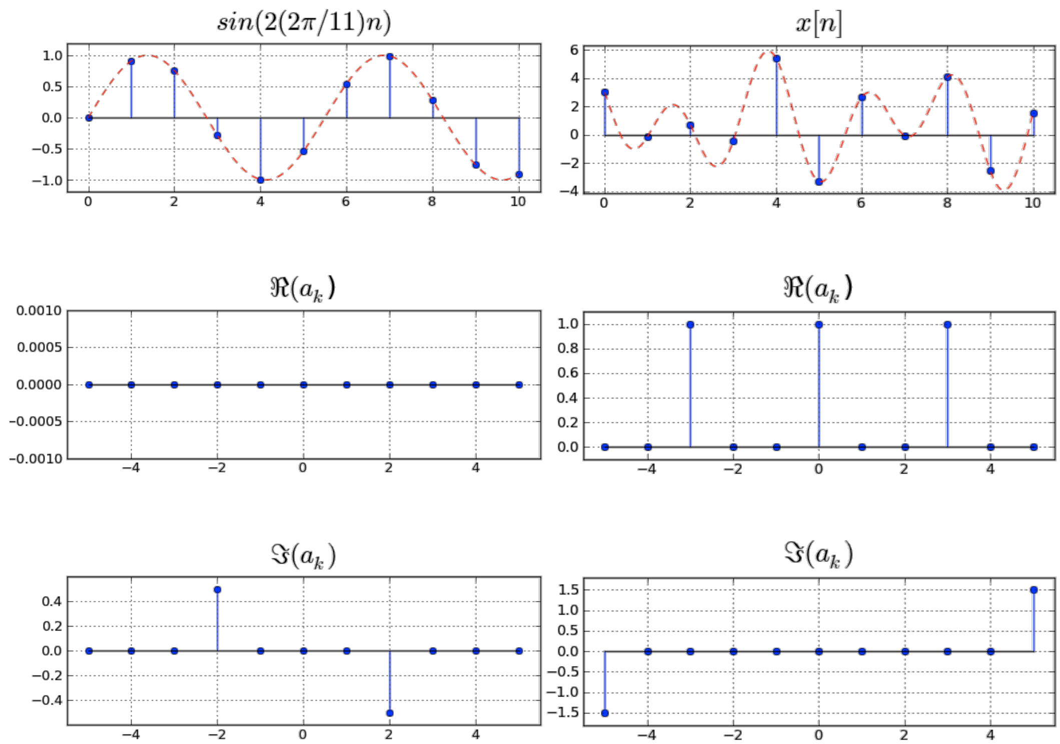

Figure 13-2: The spectrum of two periodic signals, plotted as a function of the frequency index, k, showing the real and imaginary parts for each case. $P = 11$ (odd).

direct application of Euler's identity. We turn next to a more general and systematic way of determining the spectrum for an arbitrary real P-periodic signal.

■ 13.2.2 The Analysis Equation

We now address the task of computing the spectrum of a P-periodic $x[n]$, i.e., determining the Fourier coefficients A_k. Note first that the $\{A_k\}$ comprise P coefficients that in general can be complex numbers, so in principle we have $2P$ real numbers that we can choose to match the P real values that a P-periodic real signal $x[n]$ takes in a period. It would therefore seem that we have more than enough degrees of freedom to choose the Fourier coefficients to match a P-periodic real signal. (If the signal $x[n]$ was an arbitrary *complex* P-periodic signal, hence specified by $2P$ real numbers, we would have exactly the right number of degrees of freedom.)

It turns out—and we shall prove this shortly—that for a real signal $x[n]$ the Fourier coefficients satisfy certain symmetry properties, which end up reducing our degrees of freedom to precisely P rather than $2P$. Specifically, we can show that

$$A_k = A^*_{-k} \, , \tag{13.11}$$

so the real part of A_k is an even function of k, while the imaginary part of A_k is an odd function of k. This also implies that A_0 is purely real, and also that in the case of even P, the values $A_{P/2} = A_{-P/2}$ are purely real. (These properties should remind you of the symmetry properties we exposed in connection with frequency responses in the previous chapter — but that's no surprise, because the DTFS is a similar kind of object.)

Making a careful count now of the actual degrees of freedom, we find that it takes precisely P real parameters to specify the spectrum $\{A_k\}$ for a real P-periodic signal. So given the P real values that $x[n]$ takes over a single period, we expect that Equation (13.8) will give us precisely P equations in P unknowns. (For the case of a complex signal, we will get $2P$ equations in $2P$ unknowns.)

To determine the mth Fourier coefficient A_m in the expression in Equation (13.8), where m is one of the values that k can take, we first multiply both sides of Equation (13.8) by $e^{-j\Omega_m n}$ and sum over P consecutive values of n. This results in the equality

$$\begin{aligned}
\sum_{n=\langle P\rangle} x[n]e^{-j\Omega_m n} &= \sum_{n=\langle P\rangle}\sum_{k=\langle P\rangle} A_k e^{j(\Omega_k-\Omega_m)n} \\
&= \sum_{k=\langle P\rangle} A_k \sum_{n=\langle P\rangle} e^{j\Omega_1(k-m)n} \\
&= \sum_{k=\langle P\rangle} A_k \sum_{n=\langle P\rangle} e^{j2\pi(k-m)n/P} .
\end{aligned}$$

The summation over n in the last equality involves summing P consecutive terms of a geometric series. Using the fact that for $r \neq 1$

$$1 + r + r^2 + \cdots + r^{P-1} = \frac{1-r^P}{1-r} ,$$

it is not hard to show that the above summation over n ends up evaluating to 0 for $k \neq m$. The only value of k for which the summation over n survives is the case $k = m$, for which each term in the summation reduces to 1, and the sum ends up equal to P. We therefore arrive at

$$\sum_{n=\langle P\rangle} x[n]e^{-j\Omega_m n} = A_m P$$

or, rearranging and going back to writing k instead of m,

$$A_k = \frac{1}{P}\sum_{n=\langle P\rangle} x[n]e^{-j\Omega_k n} . \tag{13.12}$$

This DTFS *analysis* equation — which holds whether $x[n]$ is real or complex — looks very similar to the DTFS synthesis equation, Equation (13.8), apart from $e^{-j\Omega_k n}$ replacing $e^{j\Omega_k n}$, and the scaling by P.

Two particular observations that follow directly from the analysis formula:

$$A_0 = \frac{1}{P}\sum_{n=\langle P\rangle} x[n] , \tag{13.13}$$

and, for the case of even P, where $\Omega_{P/2} = \pi$,

$$A_{P/2} = A_{-P/2} = \frac{1}{P} \sum_{n=\langle P \rangle} (-1)^n x[n] \; . \tag{13.14}$$

The symmetry properties of A_k that we stated earlier in the case of a real signal follow directly from this analysis equation, as we leave you to verify. Also, since $A_{-k} = A_k^*$ for a real signal, we can combine the terms

$$A_k e^{-j\Omega_k n} + A_k e^{j\Omega_k n}$$

into the single term

$$2|A_k| \cos(\Omega_k n + \angle A_k) \; .$$

Thus, for even P,

$$x[n] = A_0 + \sum_{k=1}^{P/2} 2|A_k| \cos(\Omega_k n + \angle A_k) \; ,$$

while for odd P the only change is that the upper limit becomes $(P-1)/2$.

■ 13.2.3 The Aperiodic Limit, $P \to \infty$

There is a slightly modified form in which the DTFS is sometimes written:

$$x[n] = \frac{1}{P} \sum_{k=\langle P \rangle} X_k e^{j\Omega_k n} \; , \tag{13.15}$$

which just corresponds to working with a scaled version of the A_k that we have used so far, namely

$$X_k = P A_k = \sum_{n=\langle P \rangle} x[n] e^{-j\Omega_k n} \; . \tag{13.16}$$

This form of the DTFS is useful when one considers the limiting case of *aperiodic* signals by letting $P \to \infty$, $(2\pi/P) \to d\Omega$, and $\Omega_k \to \Omega$. In this limiting case, it is easy to deduce from Equation (13.12) that $X_k \to X(\Omega)$, precisely the DTFT of the aperiodic signal that we defined in Equation (13.4). Correspondingly, the DTFS synthesis equation, Equation (13.8), in this limiting case becomes precisely the expression in Equation (13.3).

■ 13.2.4 Action of an LTI System on a Periodic Input

Suppose the input $x[\cdot]$ to an LTI system with frequency response $H(\Omega)$ is P-periodic. This signal can be represented as a weighted sum of exponentials, by the DTFS in Equation (13.8). It follows immediately that the output of the system is given by

$$y[n] = \sum_{k=\langle P \rangle} H(\Omega_k) A_k e^{j\Omega_k n} = \frac{1}{P} \sum_{k=\langle P \rangle} H(\Omega_k) X_k e^{j\Omega_k n} \; .$$

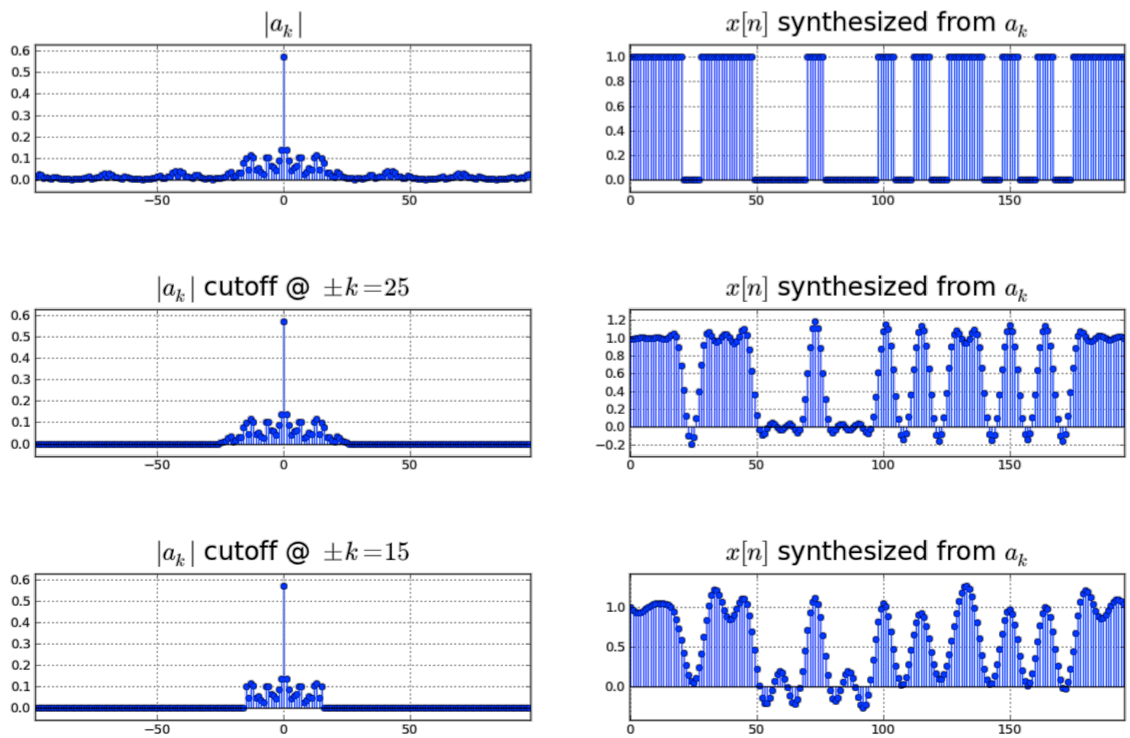

Figure 13-3: Effect of band-limiting a transmission, showing what happens when a periodic signal goes through a lowpass filter.

This immediately shows that the output $y[\cdot]$ is again P-periodic, with (scaled) spectral coefficients given by

$$Y_k = H(\Omega_k)X_k \ . \tag{13.17}$$

So knowledge of the input spectrum and of the system's frequency response suffices to determine the output spectrum. This is precisely the DTFS version of the DTFT result in Equation (13.7).

As an illustration of the application of this result, Figure 13-3 shows what happens when a periodic signal goes through an ideal lowpass filter, for which $H(\Omega) = 1$ only for $|\Omega| < \Omega_c < \pi$, with $H(\Omega) = 0$ everywhere else in $[-\pi, \pi]$. The result is that all spectral components of the input at frequencies above the cutoff frequency Ω_c are no longer present in the output. The corresponding output signal is thus more slowly varying—a "blurred" version of the input—because it does not have the higher-frequency components that allow it to vary more rapidly.

■ 13.2.5 Application of the DTFS to Finite-Duration Signals

The DTFS turns out to be useful in settings that do not involve periodic signals, but rather signals of *finite duration*. Suppose a signal $x[n]$ takes nonzero values only on some finite interval, say $[0, P-1]$ for example. We are not forbidding $x[n]$ from taking the value 0 for n within this interval, but are saying that $x[n] = 0$ for *all n outside* this interval. If we now compute the DT Fourier *transform* of this signal, according to the definition in Equation

(13.4), we get

$$X(\Omega) = \sum_{n=0}^{P-1} x[n]e^{-j\Omega n} \ . \tag{13.18}$$

The corresponding representation of $x[n]$ by a weighted combination of complex exponentials would then be the expression in Equation (13.3), involving a *continuum* of frequencies. However, it is possible to get a more economical representation of $x[n]$ by using the DT Fourier *series*.

In order to do this, consider the new signal $x_P[\cdot]$ obtained by taking the portion of $x[\cdot]$ that lies in the interval $[0, P-1]$ and replicating it periodically outside this interval, with period P. This results in $x_P[n+P] = x_P[n]$ for all n, with $x_P[n] = x[n]$ for n in the interval $[0, P-1]$. We can represent this periodic signal by its DTFS:

$$x_P[n] = \frac{1}{P} \sum_{k=<P>} X_k e^{j\Omega_k n} \ , \tag{13.19}$$

where

$$X_k = \sum_{n=<P>} x_P[n]e^{-j\Omega_k n} = \sum_{n=0}^{P-1} x[n]e^{-j\Omega_k n} \ . \tag{13.20}$$

(For consistency, we should perhaps have used the notation X_{Pk} instead of X_k, but we are trying to keep our notation uncluttered.)

We can now represent $x[n]$ by the expression in Equation (13.19), in terms of just P complex exponentials at the frequencies Ω_k defined earlier (in our development of the DTFS), rather than complex exponentials at a continuum of frequencies. However, this representation only captures $x[n]$ in the interval $[0, P-1]$. Outside of this interval, we have to ignore the expression, instead invoking our knowledge that $x[n]$ is actually 0 outside.

Another observation worth making from Equations (13.18) and (13.20) is that the (scaled) DTFS coefficients X_k are actually simply related to the DTFT $X(\Omega)$ of the finite-duration signal $x[n]$:

$$X_k = X(\Omega_k) \ , \tag{13.21}$$

so the (scaled) DTFS coefficients X_k are just P *samples* of the DTFT $X(\Omega)$. Thus any method for computing the DTFS for (the periodic extension of) a finite-duration signal will yield samples of the DTFT of this finite-duration signal (keep track of our use of DTFS versus DTFT here!). And if one wants to evaluate the DTFT of this finite-duration signal at a larger number of sample points, all that needs to be done is to consider $x[n]$ to be of finite-duration on a *larger* interval, of length $P' > P$, where of course the additional signal values in the larger interval will all be 0; this is referred to a *zero-padding*. Then computing the DTFS of (the periodic extension of) $x[n]$ for this longer interval will yield P' samples of the underlying DTFT of the signal.

As an application of the above results on finite-duration signals, consider the case of an LTI system whose unit sample response $h[n]$ is known to be 0 for all n outside of some interval $[0, n_h]$, and whose input $x[n]$ is known to be 0 for all n outside some interval $[0, n_x]$. It follows that the earliest time instant at which a nonzero output value can appear is $n = 0$, while the latest such time instant is $n = n_x + n_h$. In other words, the response $y[n] = (h * x)[n]$ is guaranteed to be 0 for all n outside of the interval $[0, n_x + n_h]$. All the interesting input/output action of the system therefore takes place for n in this interval. Outside of this interval we know that $x[\cdot]$ and $y[\cdot]$ are both 0. We can therefore take all the signals of interest to have finite duration, being 0 outside of the interval $[0, P-1]$, where $P = n_x + n_h + 1$. A DTFS representation of $x[\cdot]$ and $y[\cdot]$ on this interval, with this choice of P, can then be used to carry out a frequency-domain analysis of the system. In particular, the kth (scaled) Fourier coefficients of the input and output will be related as in Equation (13.17).

■ 13.2.6 The FFT

Implementing either the DTFS synthesis computation or the DTFS analysis computation, as defined earlier, would seem to require on the order of P^2 multiply/add operations: we have to do P multiply/adds for each of P frequencies. This can quickly lead to prohibitively expensive computations in large problems.

Happily, in 1965 Cooley and Tukey published a fast method for computing these DTFS expressions (rediscovering a technique known to Gauss!). Their algorithm is termed the **Fast Fourier Transform** or FFT, and takes on the order of $P \log P$ operations, which is a big saving. (Note that the FFT is not a new kind of transform, despite its name! — it's a fast algorithm for computing a familiar transform, namely the DTFS.)

The essence of the idea is to recursively split the computation into a DTFS computation involving the signal values at the *even* time instants and another DTFS computation involving the signal values at the *odd* time instants. One then cleverly uses the nice algebraic properties of the P complex exponentials involved in these computations to stitch things back together and obtain the desired DTFS.

The FFT has become a (or maybe *the*) workhorse of practical numerical computation. Its most common application is to computing samples of the DTFT of finite-duration signals, as described in the previous subsection. It can also be applied, of course, to computing the DTFS of a periodic signal.

■ Acknowledgments

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■ Problems and Questions

1. Let $x[\cdot]$ be a signal that is periodic with period $P = 12$. For each of the following $x[.]$, give the corresponding spectral coefficients A_k for the discrete-time Fourier series for $x[\cdot]$, for k in the range $-6 \leq k \leq 6$. (Hint: In most of the following cases, all you need to do is express the signal as the sum of appropriate complex exponentials, by inspection—this is much easier than cranking through the formal definition of the spectral coefficient.)

   (a) Determine A_k when $x[0:11] = [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0]$.

   (b) Determine A_k when $x[n] = 1$ for all n.

   (c) Determine A_k when $x[n] = sin(r(2\pi/12)n)$ for the following two choices of r:

i. $r = 3$; and
ii. $r = 8$.

(d) Determine A_k when $x[n] = sin(3(2\pi/12)n + \phi)$ where ϕ is some specified phase offset.

2. Consider a lowpass LTI communication channel with input $x[n]$, output $y[n]$, and frequency response $H(\Omega)$ given by

$$\begin{aligned} H(\Omega) &= e^{-j3\Omega} \quad \text{for } 0 \leq |\Omega| < \Omega_m \ , \\ &= 0 \qquad \text{for } \Omega_m \leq |\Omega| \leq \pi \ . \end{aligned}$$

Here Ω_m denotes the cutoff frequency of the channel; the output $y[n]$ will contain no frequency components in the range $\Omega_m \leq |\Omega| \leq \pi$. The different parts of this problem involve different choices for Ω_m.

(a) Picking $\Omega_m = \pi/4$, provide separate and properly labeled sketches of the magnitude $|H(\Omega)|$ and phase $\angle H(\Omega)$ of the frequency response, for Ω in the interval $0 \leq |\Omega| \leq \pi$. (Sketch the phase only in the frequency ranges where $|H(\Omega)| > 0$.)

(b) Suppose $\Omega_m = \pi$, so $H(\Omega) = e^{-j3\Omega}$ for *all* Ω in $[-\pi, \pi]$, i.e., *all* frequency components make it through the channel. For this case, $y[n]$ can be expressed quite simply in terms of $x[.]$; find the relevant expression.

(c) Suppose the input $x[n]$ to this channel is a periodic "rectangular-wave" signal with period 12. Specifically:

$$x[-1] = x[0] = x[1] = 1$$

and these values repeat every 12 steps, so

$$x[11] = x[12] = x[13] = 1$$

and more generally

$$x[12r - 1] = x[12r] = x[12r + 1]$$

for all integers r from $-\infty$ to ∞. At **all other times** n, we have $x[n] = 0$. (You might find it helpful to sketch this signal for yourself, e.g., for n ranging from -2 to 13.)

Find explicit values for the Fourier coefficients in the discrete-time Fourier series (DTFS) for this input $x[n]$, i.e., the numbers A_k in the representation

$$x[n] = \sum_{k=-6}^{5} A_k e^{j\Omega_k n} \ ,$$

where $\Omega_k = k(2\pi/12)$. Recall that

$$A_k = \frac{1}{12} \sum_{\langle n \rangle} x[n] e^{-j\Omega_k n} \ ,$$

where the summation is over **any** 12 **consecutive values** of n (as indicated by writing $\langle n \rangle$), so all you need to do is evaluate this expression for the particular $x[n]$ that we have.

Since $x[n]$ is an *even* function of n, all the A_k should be *purely real*, so be sure your expression for A_k makes clear that it is real. (Depending on how you proceed, you may or may not find it helpful to note that $e^{-j11(2\pi/12)} = e^{j(2\pi/12)}$.)

Check that your values for A_0 and $A_{-6} = A_6$ are correct, and be explicit about how you are checking.

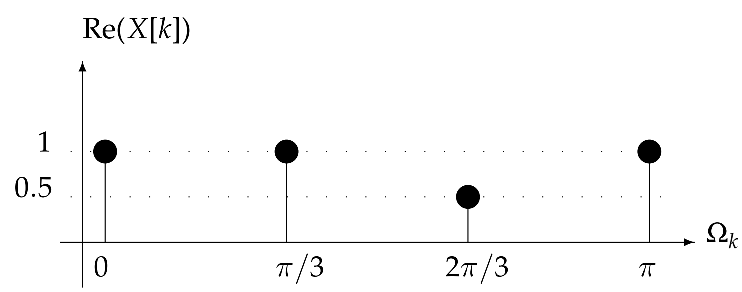
(d) Suppose the cutoff frequency of the channel is $\Omega_m = \pi/4$ (which is the case you sketched in part (a), and the input is the $x[n]$ specified in part (c). Compute the values of all the nonzero Fourier coefficients of the channel output $y[n]$, i.e., find the values of the nonzero numbers B_k in the representation

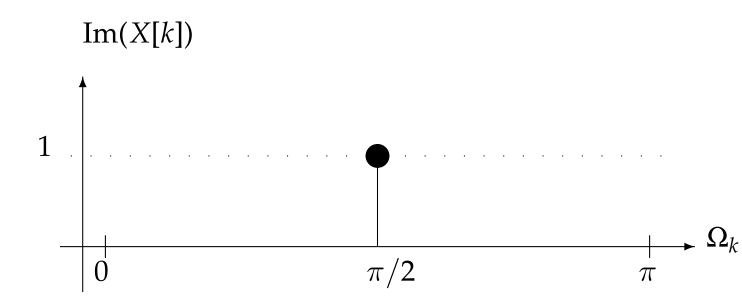
$$y[n] = \sum_{k=-6}^{5} B_k e^{j\Omega_k n} ,$$

where $\Omega_k = k(2\pi/12)$. Don't forget that $H(\Omega) = e^{-j3\Omega}$ in the passband of the filter, $0 \leq |\Omega| < \Omega_m$.

(e) Express the $y[n]$ in part (d) as an explicit and real function of time n. (If you were to sketch $y[n]$, you would discover that it is a low-frequency approximation to the $y[n]$ that would have been obtained if $\Omega_m = \pi$.)

3. The figure below shows the real and imaginary parts of all *non-zero* Fourier series coefficients $X[k]$ of a real periodic discrete-time signal $x[n]$, for frequencies $\Omega_k \in [0, \pi]$. Here $\Omega_k = k(2\pi/N)$ for some fixed even integer N, as in all our analysis of the discrete-time Fourier series (DTFS), but the plots below only show the range $0 \leq k \leq N/2$.





(a) Find all *non-zero* Fourier series coefficients of $x[n]$ at Ω_k in the interval $[-\pi, 0)$, i.e., for $-(N/2) \leq k < 0$. Give your answer in terms of careful and fully labeled plots of the real and imaginary parts of $X[k]$ (following the style of the figure above).

(b) Find the period of $x[n]$, i.e., the smallest integer T for which $x[n+T] = x[n]$, for all n.

(c) For the frequencies $\Omega_k \in [0, \pi]$, find all non-zero Fourier series coefficients of the signal $x[n-6]$ obtained by delaying $x[n]$ by 6 samples.