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6.02 **Solutions to Chapter 12** Updated: April 5, 2012

Please send information about errors or omissions to hari; questions are best asked on piazza.

1. (a) From the definition of the frequency response, $H(\Omega) = \sum_{m=\infty}^{\infty} h[m]e^{-j\Omega m} = 1 + 2e^{-j\Omega} + e^{-2j\Omega}$.

   (b) Plug in the given Ω values into the expression for $H(\Omega)$, to get $H(0) = 1, H(\pi/2) = 1 - 2j - 1 = -2j, H(\pi) = 1 - 2 + 1 = 0$.

   (c) Let $e^{-j\Omega} = z$, so we want $1 + 2z + z^2 = 0$, which gives us $(z+1)^2 = 0$, or $z = -1$. Hence, $e^{-j\Omega} = -1$, or $\Omega = \pi$.

2. (a) We'll use the fact that if the input to an LTI with frequency response $H(\Omega)$ is a complex exponential $x[n] = e^{j\Omega n}$, then the output $y[n] = H(\Omega)x[n] = H(\Omega)e^{j\Omega n}$. Plugging that into the given equation or $y[n]$, we get $H(z) = 1 + \alpha z + \beta z^2 + \gamma z^3$, denoting $e^{-j\Omega}$ by z for convenience (on both sides of the equation).

   Let's now expand the given expression, $H(\Omega) = 1 - 0.5e^{-j2\Omega}\cos\Omega$ in terms of complex exponentials. The idea is we can then match the coefficients to obtain the values of α, β, and γ. Expanding the $\cos(\Omega)$ term into complex exponential sums and multiplying, we get the expression $H(z) = 1 - 0.25z - 0.25z^3$. Matching coefficients, we conclude that $\alpha = -0.25, \beta = 0, \gamma = -0.25$.

   (b) For a series (cascade) of LTI systems, the frequency response is the *product* of the frequency responses of the individual LTI systems. So, the frequency response of this cascade is $G(e^{j\Omega})H(e^{j\Omega})$. For $w[n]$ to be equal to $x[n]$ for all n, the unit sample response of the cascade must be $\delta[n]$. The frequency response of the cascade is related the unit sample response by $G(e^{j\Omega})H(e^{j\Omega}) = \sum_{m=0}^{m=\infty} \delta[m]e^{-j\Omega m} = 1$. Since $G(e^{j\Omega})H(e^{j\Omega}) = 1$, and $H(e^{j\Omega})$ is never zero for $-\pi \le \Omega \le \pi$, then $G(e^{j\Omega}) = \frac{1}{H(e^{j\Omega})}$. Hence,

   $$G(e^{j\Omega}) = \frac{1}{1 - 0.5e^{-j2\Omega}\cos\Omega)}.$$

   (c) First, observe that we can rewrite the given $x[n]$ as

   $$x[n] = e^{j0n} + 0.5e^{j\pi n}.$$

   Then using the meaning of frequency response, and applying superposition, we can write

   $$y[n] = 0e^{j0n} + Ae^{j\pi n} = H(e^{j0})e^{j0n} + H(e^{j\pi n})0.5e^{j\pi n}.$$

   Matching terms, it then follows that $H(e^{j0}) = 0$ and $A = 0.5H(e^{j\pi n})$. For this difference equation, the frequency response for an arbitrary Ω is given by

   $$H(e^{j\Omega}) = 1 + \alpha e^{-j\Omega} + \beta e^{-j2\Omega} + \gamma e^{-j3\Omega},$$

so for the special case of $\Omega = 0$,

$$H(e^{j0}) = 1 + \alpha + \beta + \gamma.$$

Given that $H(e^{j0}) = 0$, and $\alpha = \gamma = 1$, it must be true that $\beta = -3$. Then to determine A, consider that

$$H(e^{j\pi n}) = 1 + \alpha e^{-j\pi} + \beta e^{-j2\pi} + \gamma e^{-j3\pi} = 1 - \alpha\ + \beta - \gamma = 1 - 1 - 3 - 1 = -4.$$

Therefore, $A = 0.5H(e^{j\pi n}) = 0.5(-4) = -2$.

3. (a) From the definition of the frequency response, $H(\Omega) = \sum_{m=0}^{\infty} h[m]e^{-j\Omega m} = h[0]e^{-j\Omega 0} + h[1]e^{-j\Omega} + h[2]e^{-j2\Omega} + h[3]e^{-j3\Omega}$. Plugging in values for $h[i]$, the frequency response is $H(\Omega) = a + be^{-j\Omega} + be^{-j2\Omega} + ae^{-j3\Omega}$.

   (b) Note that $x[n] = (-1)^n$ can be written as $x[n] = e^{-j\pi n}$. This input corresponds to a complex exponential with angular frequency $\Omega = \pi$. We know that for everlasting complex exponential inputs, $y[n] = H(\Omega)e^{j\Omega n}$; therefore we know that $y[n] = H(\pi)e^{j\pi n}$. Evaluating $H(\Omega)$, $y[n]$ is given by $(a - b + b - a)$, or $y[n] = 0$ for all n.

   (c) The definition of convolution follows as $y[n] = \sum_{m=0}^{\infty} h[m]x[n-m]$. We find $y[5]$ by solving $\sum_{m=0}^{\infty} h[m]x[5-m]$; this gives $-a + b - b + a = 0$. Similarly, $y[6] = \sum_{m=0}^{\infty} h[m]x[6-m]$, giving $a - b + b - a = 0$. Moreover, since these calculations are representative of what one would get for odd and even n respectively, $y[n] = 0$ for all n.

   (d) Decompose into cosines. Dividing both sides by $e^{-j3\Omega 2}$ we know that $G(\Omega) = \frac{1}{e^{-j3\Omega/2}}H(\Omega)$. Dividing out $e^{-j3\Omega/2}$ from each term in $H(\Omega)$, we get $ae^{j3\Omega/2} + be^{j\Omega/2} + be^{-j\Omega/2} + ae^{j3\Omega/2}$. Converting this expression to cosines, we find that

$$G(\Omega) = 2[a\cos(\frac{3\Omega}{2}) + b\cos(\frac{\Omega}{2})].$$

   (e) By superposition, the response is the sum of the responses to $(-1)^n$ and to $\cos(\frac{\pi}{2}n + \theta_0)$. But we have already seen in parts (b) and (c) that the response to $(-1)^n$ is 0 for all n. So what remains is to find the response to $\cos(\pi n + \theta_0)$. This response is simply

$$y[n]] = |H(\pi/2)|\cos(\frac{\pi}{2}n + \theta_0 + \angle H(\pi/2)).$$

   Note from the form of $H(\Omega)$ in part (d) that

$$\begin{aligned} H(\pi/2) = \quad & G(\pi/2)e^{-j3\pi/4} = 2[a\cos(\frac{3\pi}{4}) + b\cos(\frac{\pi}{4})] \\ = \quad & \sqrt{2}(b-a)e^{-j3\pi/4} \end{aligned}$$

   Since $b > a$, it follows that $|H(\pi/2)| = \sqrt{2}(b-a)$ and $\angle H(\pi/2) = -3\pi/4$. Hence,

$$y[n] = \sqrt{2}(b-a)\cos\Big(\frac{\pi}{2}n + \theta_0 - \frac{3\pi}{4}\Big).$$

   As a check, consider the case $\theta_0 = 0$, so the input of interest is $\cos(\pi n)$, which alternates between $+1$ and -1 at even values of n, and is 0 at all odd values of n. Convolving this sequence with the given unit sample response $h[n]$ shows that $y[0] = a - b$. And this answer matches what we get on evaluating the above general expression for $y[n]$ at $n = 0$ and $\theta_0 = 0$.

4. (a) There are two key things to notice about this frequency response function. First off, it has no zeros. Second, looking at the denmoinator, when $\Omega \approx \pm\frac{\pi}{2}$ the denominator becomes very small, causing $H_A(e^{j\Omega})$ to become large. The only graph that satisfies these two criteria is H_I.

   (b) We can solve this problem by setting $x[n] = e^{j\Omega}$ and knowing that the frequency response is a complex exponential with the same frequency. We get

$$H(\Omega)e^{j\Omega n} + a_1 H(\Omega)e^{j\Omega(n-1)} + a_2 H(\Omega)e^{j\Omega(n-2)} = e^{j\Omega n}.$$

Dividing both sides by $e^{j\Omega n}$ and solving for $H(\Omega)$, we get

$$H(\Omega) = \frac{1}{1 + a_1 e^{-j\Omega} + a_2 e^{-j2\Omega}}.$$

It is easier to expand $H_A(\Omega)$ than to factor $H(\Omega)$, so we factor $H_A(\Omega)$ to find the denominator is $1 + 0.95e^{j(\Omega+\frac{\pi}{2})} + 0.95e^{j(\Omega-\frac{\pi}{2})} + 0.95e^{j(\Omega-\frac{\pi}{2})}0.95e^{j(\Omega+\frac{\pi}{2})}$. The middle two terms cancel each other out (because of the $\frac{pi}{2}$ phase shifts) and the last term is equal to $0.9025e^{-j2\Omega}$. Matching coefficients with $H(\Omega)$, we find that $a_1 = 0$ and $a_2 = \left(\frac{19}{20}\right)^2 = 0.9025$.

5. (a) We can make the following observations:

$$\begin{aligned}
\max_n(cos\frac{\pi}{6}n) &= 1, n = 0, 12, \ldots \\
\max_n(cos\frac{5\pi}{6}n) &= 1, n = 0, 12, \ldots \\
\max_n(3(-1)^n) &= 1, n \text{ even.}
\end{aligned}$$

Hence, the maximum value is 7, and the smallest *positive* n at which the maximum occurs is $n = 12$.

   (b) The frequency response at $\Omega = \frac{\pi}{6}$ and at $\Omega = \frac{5\pi}{6}$ must be zero, which means that the only possibility is H_{III}.

$$y[n] = H(0) \cdot 2 \cdot (1)^n + H(\pi) \cdot 3 \cdot (-1)^n = 4 \cdot 2 \cdot (1)^n + 4 \cdot 3 \cdot (-1)^n.$$

The numerical value of M is 4.

6. (a) The first equation tells us that the DC component of the frequency response, i.e., $H(0)$ is 5. The second and third equations tell is that $H(\pi/2)$ and $H(-\pi/2)$ are both 0.

   (b) H_{II} is the frequency response that best describes the above information, as it is the only curve that meets the constraints of part (a). The numerical value of M must be 5.

   (c) $y[n]/x[n]$ is the frequency response $H(\Omega)$. Because the input is an everlasting exponential with frequency $\Omega = \frac{\pi}{6}$, $y[n]/x[n]$ is simply $H(\frac{\pi}{6}) = h[0] + h[2]e^{-j\pi/3} = 3.75 - j(5\sqrt{3}/4)$.