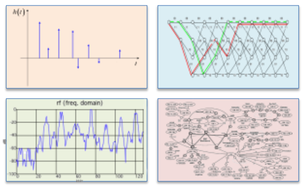


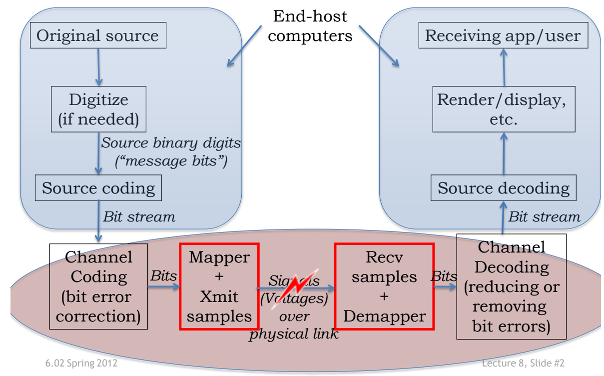
INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Spring 2012
Lecture #8

- Noise: bad things happen to good signals!
- Additive white Gaussian noise (AWGN)
- Bit error rate analysis
- Signal-to-noise ratio and decibel (dB) scale
- Binary symmetric channel (BSC) abstraction

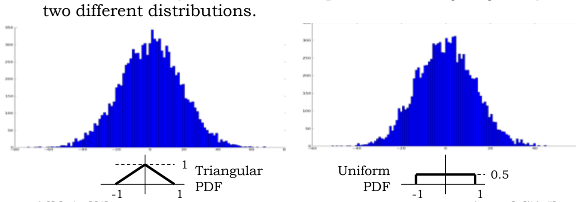
Single Link Communication Model



Noise on a Communication Channel

The net noise observed at the receiver is often the sum of many small, independent random contributions from many factors. If these independent random variables have finite mean and variance, the Central Limit Theorem says their sum will be a *Gaussian*.

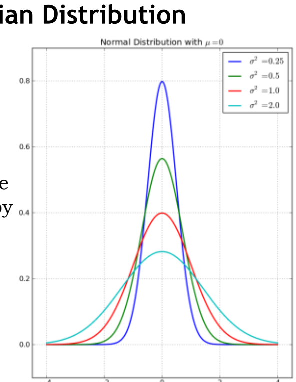
The figure below shows the histograms of the results of 10,000 trials of summing 100 random samples draw from [-1,1] using two different distributions.



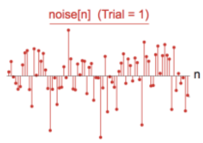
The Gaussian Distribution

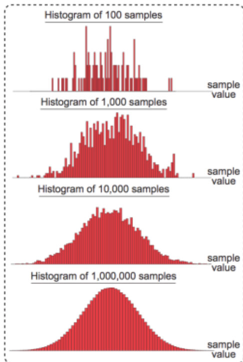
A Gaussian distribution with mean μ and variance σ² has a PDF described by

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$



From Histogram to PDF





Experiment: create histograms of sample values from trials of increasing lengths.

If distribution is stationary, then histogram converges to a shape known as a probability density function (PDF)

Estimating noise

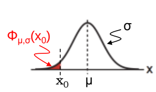
- Transmit a sequence of "0" bits, i.e., hold the voltage V_0 at the transmitter
- Observe received samples y[k], k = 0, 1, . . . , K − 1
  - Process these samples to obtain the statistics of the noise process for additive noise. Under the assumption of no distortion, and constant (or "stationary") noise statistics,
- Noise samples w[k] = y[k] − V_0
- For large K, can use the sample mean m to estimate μ, where

$$m = \frac{1}{K}\sum_{k=0}^{K-1} w[k] \qquad s^2 = \frac{1}{K}\sum_{k=0}^{K-1} (w[k] - m)^2$$

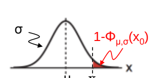
Cumulative Distribution Function

When analyzing the effects of noise, we'll often want to determine the probability that the noise is larger or smaller than a given value x_0.

$$p(x \le x_0) = \int_{-\infty}^{x_0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx \equiv \Phi_{\mu,\sigma}(x_0)$$



$$p(x \ge x_0) = \int_{x_0}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = 1 - \Phi_{\mu,\sigma}(x_0)$$



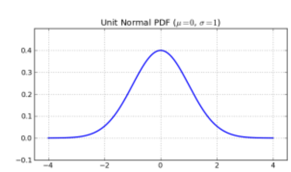
Where $\Phi_{\mu,\sigma}(x)$ is the cumulative distribution function (CDF) for the normal distribution with mean μ and variance σ^2. The CDF for the unit normal is usually written as just Φ(x).

$$\Phi_{\mu,\sigma}(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Φ(x) = CDF for Unit Normal PDF

Most math libraries don't provide Φ(x) but they do have a related function, erf(x), the *error function*:

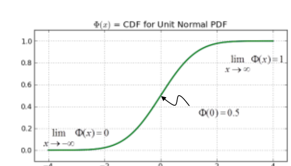
$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$



For Python hackers:

```
from math import sqrt
from scipy.special import erf

# CDF for Normal PDF
def Phi(x,mu=0,sigma=1):
  t = erf((x-mu)/(sigma*sqrt(2)))
  return 0.5 + 0.5*t
```



CDF and erfc

$$erf(w) = \frac{2}{\sqrt{\pi}}\int_0^w e^{-t^2}dt = 2\frac{1}{\sqrt{2\pi}}\int_0^{w\sqrt{2}} e^{-\frac{x^2}{2}}dx$$

$$\Phi(x_0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_0} e^{-\frac{x^2}{2}}dx = 0.5 + \frac{1}{\sqrt{2\pi}}\int_0^{x_0} e^{-\frac{x^2}{2}}dx$$

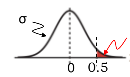
$$\Phi(x_0) = 0.5 + 0.5erf(\frac{x_0}{\sqrt{2}}) = 0.5 - 0.5erf(-\frac{x_0}{\sqrt{2}}) = 0.5erfc(-\frac{x_0}{\sqrt{2}})$$

$$\Phi_{\mu,\sigma}(x) = \Phi(\frac{x-\mu}{\sigma}) = 0.5 + 0.5erf(\frac{x-\mu}{\sigma\sqrt{2}}) = 0.5erfc(-\frac{x-\mu}{\sigma\sqrt{2}})$$

p(bit error)

Now assume the channel has Gaussian noise with μ=0 and variance σ^2. And we'll assume a digitization threshold of 0.5V. We can calculate the probability that noise[k] is large enough that y[k] = y_{nf}[k] + noise[k] is received incorrectly:

p(error | transmitted "0"):

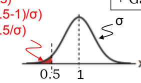


$1-\Phi_{\mu,\sigma}(0.5) = \Phi_{\mu,\sigma}(-0.5)$
$= \Phi((-0.5-0)/\sigma)$
$= \Phi(-0.5/\sigma)$

Plots of noise-free voltage + Gaussian noise

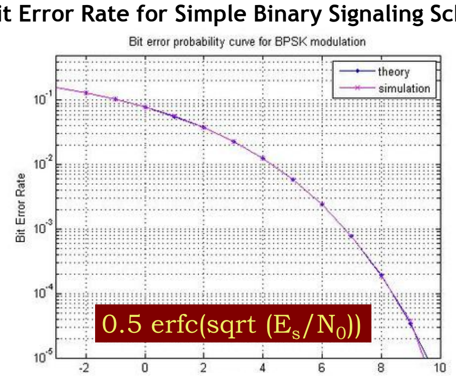
p(error | transmitted "1"):

$\Phi_{\mu,\sigma}(0.5)$
$= \Phi((0.5-1)/\sigma)$
$= \Phi(-0.5/\sigma)$



p(bit error) = p(transmit "0")*p(error | transmitted "0") + p(transmit "1")*p(error | transmitted "1")
= 0.5*Φ(-0.5/σ) + 0.5*Φ(-0.5/σ)
= Φ(-0.5/σ)

Bit Error Rate for Simple Binary Signaling Scheme



http://www.dsplog.com/2007/08/05/bit-error-probability-for-bpsk-modulation/

Signal-to-Noise Ratio (SNR)

The Signal-to-Noise ratio (SNR) is useful in judging the impact of noise on system performance:

$$\text{SNR} = \frac{\tilde{P}_{signal}}{\tilde{P}_{noise}}$$

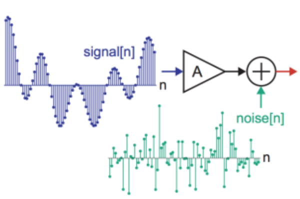
SNR is often measured in decibels (dB):

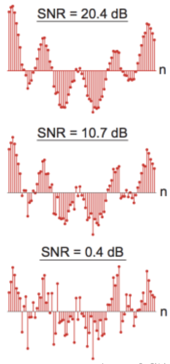
$$\text{SNR (db)} = 10\log\left(\frac{\tilde{P}_{signal}}{\tilde{P}_{noise}}\right)$$

3db is a factor of 2

10logX	X
100	10000000000
90	1000000000
80	100000000
70	10000000
60	1000000
50	100000
40	10000
30	1000
20	100
10	10
0	1
-10	0.1
-20	0.01
-30	0.001
-40	0.0001
-50	0.000001
-60	0.0000001
-70	0.00000001
-80	0.000000001
-90	0.0000000001
-100	0.00000000001

SNR Example





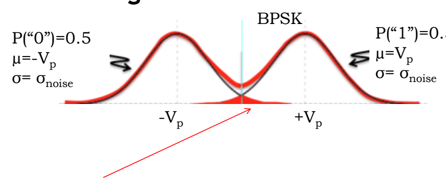
Changing the amplification factor (gain) A leads to different SNR values:

- Lower A → lower SNR
- Signal quality degrades with lower SNR

Connecting the SNR and BER

$V_p = \sqrt{E_S}$

$2\sigma^2 = N_0$



$$\text{BER} = \mathbb{P}(\text{error}) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\sqrt{E_s}}^{\infty} e^{-w^2/(2\sigma^2)}\, dw$$

$$erf(w) = \frac{2}{\sqrt{\pi}} \int_0^w e^{-t^2} dt$$

$$erfc(w) = 1 - erf(w) = \frac{2}{\sqrt{\pi}} \int_w^\infty e^{-x^2} dx$$

$$\text{BER} = \mathbb{P}(\text{error}) = \frac{1}{\sqrt{\pi}} \cdot \int_{\sqrt{E_s/N_0}}^{\infty} e^{-v^2}\, dv$$

$$BER = P(error) = \frac{1}{2} erfc(\sqrt{\frac{E_S}{N_0}}) = \frac{1}{2} erfc(\sqrt{\frac{SNR}{2}}) = \frac{1}{2} erfc(\frac{V_p}{\sigma\sqrt{2}})$$

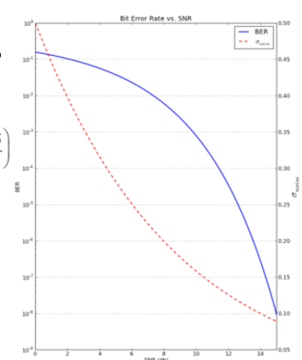
BER vs. SNR

For V_p=0.5, we calculate the power of the noise-free signal to be 0.25 and the power of the Gaussian noise is its variance, so

$$\text{SNR (db)} = 10 \log \left(\frac{\tilde{P}_{signal}}{\tilde{P}_{noise}} \right) = 10 \log \left(\frac{0.25}{\sigma^2} \right)$$

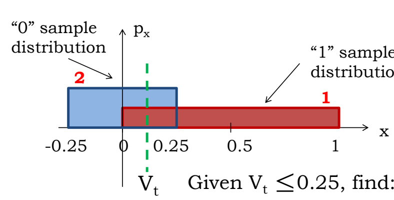
Given an SNR, we can use the formula above to compute σ^2 and then plug that into the formula on the previous slide to compute p(bit error) = BER.

The BER result is plotted to the right for various SNR values.



Spot quiz

p("0")=p("1")=0.5



Given $V_t \leq 0.25$, find:

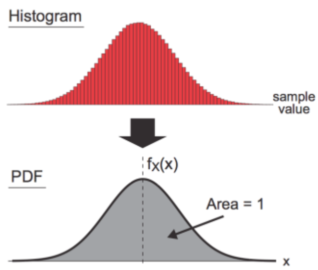
P(1 received | 0 sent) = 2*(0.25 – Vt)
P(0 received | 1 sent) = 1*(Vt-0)=Vt
P(error) = 0.5*2*(0.25-Vt)+0.5*Vt =0.25-0.5*Vt
Value of V_t that minimizes P(error): V_t= 0.25
Value of min P(error): P(error) $_{min}$= 0.25*0.5 = 0.125

Formalizing the PDF Concept

Define x as a random variable whose PDF has the same shape as the histogram we just obtained.

Denote the PDF of x as $f_x(x)$ and scale $f_x(x)$ such that its overall area is 1:

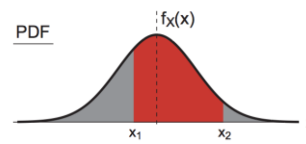
$$\int_{-\infty}^{\infty} f_x(x)dx = 1$$



Formalizing Probability

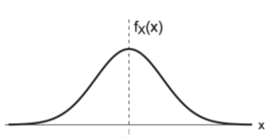
The probability that random variable x takes on a value in the range of x_1 to x_2 is calculated from the PDF of x as:

$$p(x_1 \le x < x_2) = \int_{x_1}^{x_2} f_x(x)dx$$



A PDF is NOT a probability – its integral is.
Note that probability values are always in the range of 0 to 1.

Mean and Variance



The *mean* of a random variable x, μ_x, corresponds to its average value and computed as:

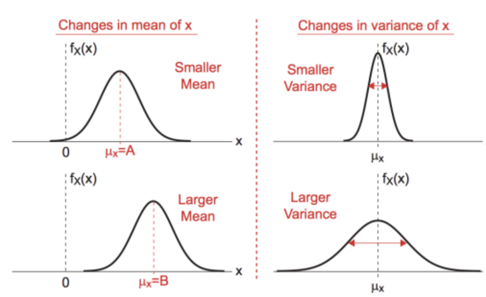
$$\mu_x = \int_{-\infty}^{\infty} x\, f_x(x)dx$$

The *variance* of a random variable x, σ_x^2, gives an indication of its variability and is computed as:

Compare with power calculation

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-\mu_x)^2 f_x(x)dx$$

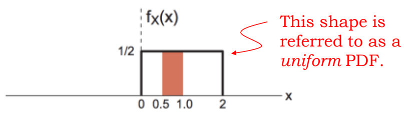
Visualizing Mean and Variance



Changes in mean shift the center of mass of PDF

Changes in variance narrow or broaden the PDF (but area is always equal to 1)

Example Probability Calculation



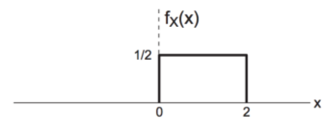
Verify that overall area is 1:

$$\int_{-\infty}^{\infty} f_x(x)dx = \int_0^2 0.5\,dx = 1$$

Probability that x takes on a value between 0.5 and 1:

$$p(0.5 \le x \le 1.0) = \int_{0.5}^{1} 0.5\,dx = 0.25$$

Example Mean and Variance Calculation



Mean:

$$\mu_x = \int_{-\infty}^{\infty} x\, f_x(x)dx = \int_0^2 x\frac{1}{2}dx = \frac{1}{4}x^2\Big|_0^2 = 1$$

Variance:

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-\mu_x)^2 f_x(x)dx = \int_0^2 (x-1)^2 \frac{1}{2}dx = \frac{1}{6}(x-1)^3\Big|_0^2 = \frac{1}{3}$$