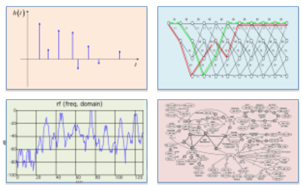
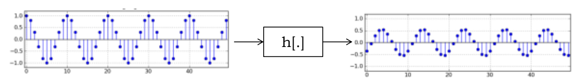


INTRODUCTION TO EECS II
DIGITAL COMMUNICATION SYSTEMS

6.02 Spring 2012
Lecture #13

- Frequency Response of LTI systems
- Building better filters

Sinusoidal Inputs to LTI Systems



Sinusoidal inputs, i.e.,

$$x[n] = \cos(\Omega n + \theta)$$

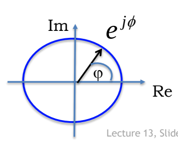
yield sinusoidal outputs at the same 'frequency' Ω rads.

Complex Exponentials

$$e^{j\varphi} = \cos(\varphi) + j\sin(\varphi)$$

$$\cos(\varphi) = \frac{1}{2}e^{j\varphi} + \frac{1}{2}e^{-j\varphi} \qquad \sin(\varphi) = \frac{1}{2j}e^{j\varphi} - \frac{1}{2j}e^{-j\varphi}$$

In the complex plane, $e^{j\varphi} = \cos(\varphi) + j\sin(\varphi)$ is a point on the unit circle, at an angle of φ with respect to the positive real axis. Increasing φ by 2π brings you back to the same point! So any function of $e^{j\varphi}$ only needs to be studied for φ in [-π, π] .



Frequency Response

$Ae^{j\Omega n}$ → h[.] → y[n]

Using the convolution sum we can compute the system's response to a complex exponential (of frequency Ω) as input:

$$y[n] = \sum_m h[m]x[n-m]$$
$$= \sum_m h[m]Ae^{j\Omega(n-m)}$$
$$= \left(\sum_m h[m]e^{-j\Omega m}\right)Ae^{j\Omega n}$$
$$= H(\Omega)\cdot x[n]$$

where we've defined the *frequency response* of the system as

$$H(\Omega) \equiv \sum_m h[m]e^{-j\Omega m}$$

This is an infinite sum in general, but is well behaved if h[.] is absolutely summable, i.e., if the system is stable.

From Complex Exponentials to Sinusoids

$\cos(\Omega n) = (e^{j\Omega n} + e^{-j\Omega n})/2$

So response to this cosine input is

$(H(\Omega)e^{j\Omega n} + H(-\Omega)e^{-j\Omega n})/2$ = Real part of $H(\Omega)e^{j\Omega n}$

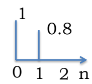
= Real part of $|H(\Omega)|e^{j(\Omega n + \angle H(\Omega))}$

$\cos(\Omega_0 n)$ → H(Ω) → $|H(\Omega_0)|\cos(\Omega_0 n + \angle H(\Omega_0))$

Example: Channel with Echo

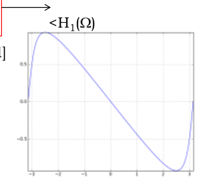
x[n] → LTI Channel $h_1[.]$ → y[n]

$h_1[n] = \delta[n] + 0.8\delta[n-1]$



$H_1(\Omega) = ??$

$$H_1(\Omega) = \sum_m h_1[m] e^{-j\Omega m}$$



$H_1(\Omega) = h_1[0] + h_1[1]e^{-j\Omega} = 1 + 0.8e^{-j\Omega} = 1 + 0.8\cos(\Omega) - j0.8\sin(\Omega)$

So:

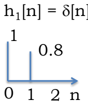
$|H_1(\Omega)| = [1.64 + 1.6\cos(\Omega)]^{1/2}$ *EVEN function of* Ω;

$\angle H_1(\Omega) = \arctan[-(0.8\sin(\Omega)/[1 + 0.8\cos(\Omega)]$ *ODD*.

Example: Channel with Echo

x[n] → LTI Channel $h_1[.]$ → y[n]

$h_1[n] = \delta[n] + 0.8\delta[n-1]$



What is y[n] if $x[n] = 0.5\cos(n\pi/2) + \sin(n\pi)$?

$\cos(\Omega_0 n)$ -> $|H(\Omega_0)|\cos(\Omega_0 n + \angle H(\Omega_0))$

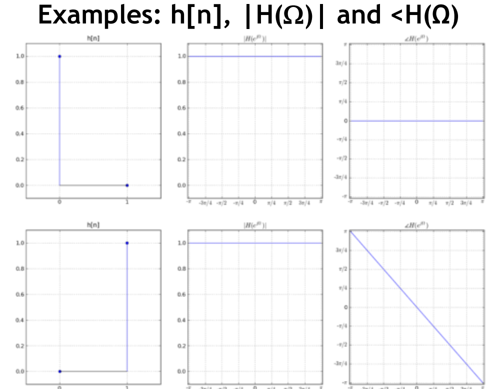
$|H_1(\Omega)| = [1.64 + 1.6\cos(\Omega)]^{1/2}$

$\angle H_1(\Omega) = \arctan[-(0.8\sin(\Omega)/[1 + 0.8\cos(\Omega)]$

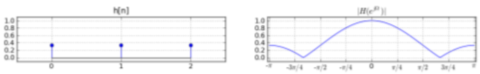
$y[n] = 0.5|H_1(\pi/2)|\cos(n\pi/2 + \angle H_1(\pi/2)) + |H_1(\pi)|\sin(n\pi + \angle H_1(\pi))$

$y[n] = 0.64\cos(n\pi/2 - 0.67) + 0.2\sin(n\pi)$

Examples: h[n], |H(Ω)| and <H(Ω)



H(Ω) with Zeros



$$H(\Omega)=\sum_m h[m]e^{-j\Omega m}=h[0]e^{-j\Omega 0}+h[1]e^{-j\Omega 1}+h[2]e^{-j\Omega 2}$$
$$=h[0]+h[1](e^{-j\Omega})+h[2](e^{-j\Omega})^2$$

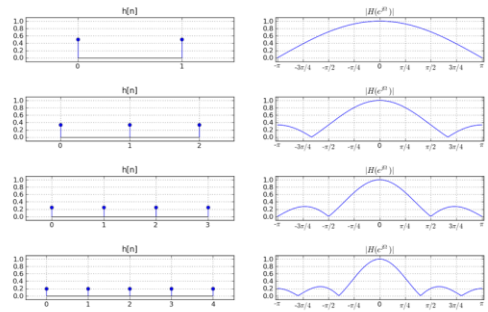
Hmm. A quadratic equation with two roots at $\Omega=\pm\varphi$:

$$(e^{-j\Omega}-e^{-j\varphi})(e^{-j\Omega}-e^{j\varphi})$$
$$=(e^{-j\Omega})^2-(e^{j\varphi}+e^{-j\varphi})(e^{-j\Omega})+e^{j\varphi}e^{-j\varphi}$$
$$=1-2\cos(\varphi)(e^{-j\Omega})+(e^{-j\Omega})^2$$

Matching terms in the two equations, we see that this LTI system would have a frequency response that went to zero at $\pm\varphi$ if

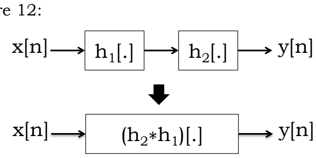
h[0]=1, h[1]=−2cos(φ) and h[2] = 1.

Frequency Response of "Moving Average" Filters

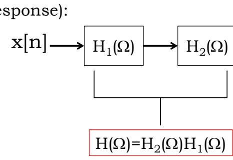


Series Interconnection of LTI Systems

From Lecture 12:



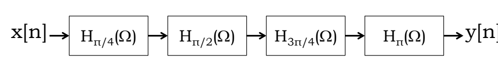
In the frequency domain (i.e., thinking about input-to-output frequency response):

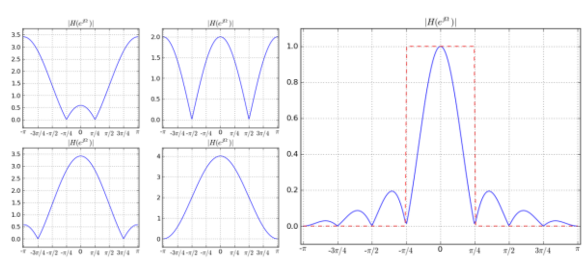


i.e., convolution in time has become multiplication in frequency!

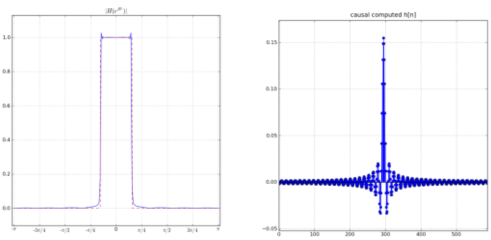
A 10-cent Low-pass Filter

Suppose we wanted a low-pass filter with a cutoff frequency of π/4





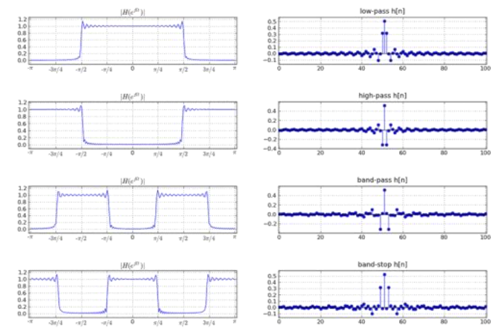
The $4.99 version, h[n] and H(Ω)



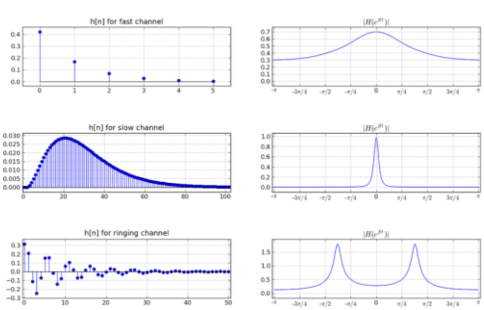
$H(\Omega) = 1 \quad |\Omega| < \Omega_c,$
$H(\Omega) = 0 \quad \Omega_c < |\Omega| \le \pi$

$$h[n] = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega n}\,d\Omega = \begin{cases} \frac{\sin(\Omega_c n)}{\pi n} & \text{for } n \neq 0 \\ \frac{\Omega_c}{\pi} & \text{for } n = 0 \end{cases}$$

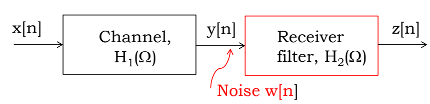
H(Ω) and h[n] for some Useful Filters



h[n] and H(Ω) for some Idealized Channels



A Frequency-Domain view of Deconvolution



Given $H_1(\Omega)$, what should $H_2(\Omega)$ be, to get z[n]=x[n]?

⇨ $H_2(\Omega)=1/H_1(\Omega)$ "Inverse filter"

$= (1/|H_1(\Omega)|).\ \exp\{-j\angle H_1(\Omega)\}$

Inverse filter at receiver does very badly in the presence of noise that adds to y[n]:

filter has high gain for noise precisely at frequencies where channel gain $|H_1(\Omega)|$ is low (and channel output is weak)!