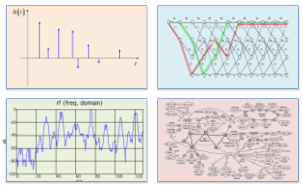


INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

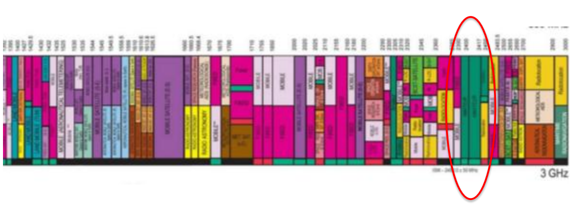
6.02 Spring 2012
Lecture #14

- Viewing signals in frequency domain
- Discrete-time Fourier Series

U.S. Spectrum Allocation Map

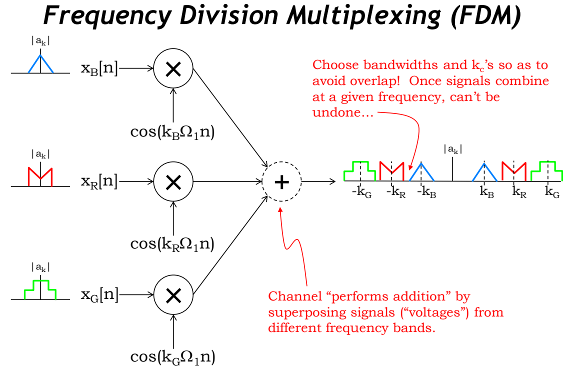
http://www.wireless-technology.org/wp-content/uploads/2011/02/Wireless-Spectrum-Photo.jpg

U.S. Spectrum Allocation Map

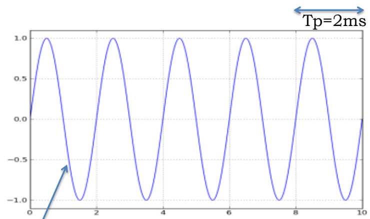


http://www.wireless-technology.org/wp-content/uploads/2011/02/Wireless-Spectrum-Photo.jpg

Multiple Transmitters: *Frequency Division Multiplexing (FDM)*



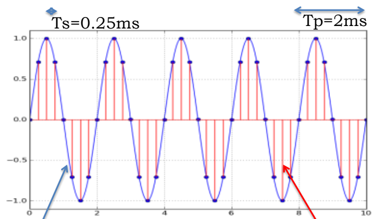
Continuous and Discrete-time signals



Example
500Hz
carrier tone

$\sin(\frac{2\pi}{T_p}t)$ **(Continuous-time sinewave)**

Continuous and Discrete-time signals



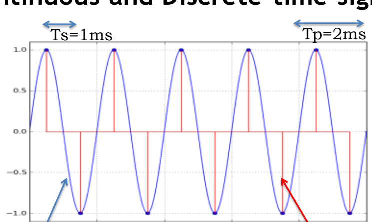
$\sin(\frac{2\pi}{T_p}t)$ **(Continuous-time sinewave)**

$\sin(\frac{2\pi}{P}n)$

$\Omega_0 = \frac{2\pi}{T_p}T_s = \frac{2\pi}{P} = \frac{2\pi}{8} = \frac{\pi}{4}$ **(Discrete-time sinewave)**

P= T_p/T_s = 8 samples per period

Continuous and Discrete-time signals



Example
500Hz
carrier tone

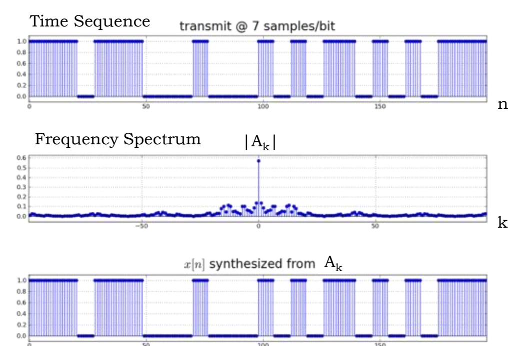
$\sin(\frac{2\pi}{T_p}t)$ **(Continuous-time sinewave)**

$\sin(\frac{2\pi}{P}n)$

$\Omega_0 = \frac{2\pi}{T_p}T_s = \frac{2\pi}{P} = \frac{2\pi}{2} = \pi$ **(Discrete-time sinewave)**

P= T_p/T_s = 2 samples per period

Spectrum of Digital Transmissions



Representing Periodic Signals in Frequency Domain

Any *strictly* periodic DT signal of period **P**

x[n+P]=x[n] for all n

e.g., 6.sin((2πn/P)+0.17) + 4.cos(3(2πn/P)+0.82)

can be written as
a weighted combination (generally with complex weights)
of **P** complex exponentials

whose frequencies are
consecutive integer multiples of the *fundamental frequency* $\mathbf{2\pi/P=\Omega_1}$
(so each exponential term has period P)

This is the
Discrete-Time Fourier Series (DTFS)
or discrete spectral representation.

Discrete-Time Fourier Series (DTFS)

If x[n] is periodic with period P (convenient to assume P is even, so P/2 is integer, but odd P can be handled too), it can be expressed as the sum of P "spectral components" --- scaled complex exponentials of period P:

$$x[n] = \sum_{k=\langle P \rangle} A_k e^{jk\Omega_1 n}$$

Complex exponentials with *fundamental frequency* 2π/P = Ω_1. Frequency of term k is $\Omega_k = k\Omega_1$.

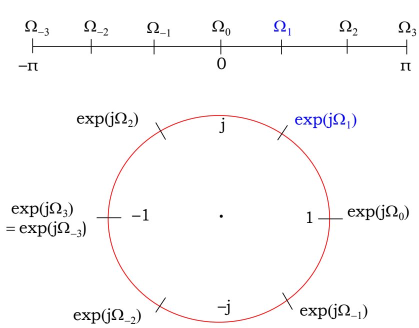
k ranges over any P consecutive integers. Common choices:

- k for 0 to P−1 ; $0 \le k\Omega_1 \le 2\pi - \Omega_1$
- k for −(P/2) to (P/2)−1 for even P ; $-\pi \le k\Omega_1 \le \pi - \Omega_1$
- k symmetrically out from 0 for odd P ; $-\pi + (\Omega_1/2) \le k\Omega_1 \le \pi - (\Omega_1/2)$

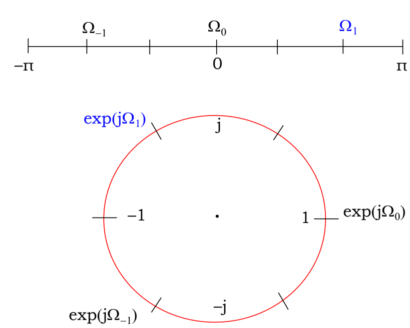
$$= \frac{1}{P} \sum_{k=\langle P \rangle} X_k e^{jk\Omega_1 n}$$

With the notation $A_k = X_k/P$, we get an alternate (and often used) normalization.

Where do the Ω_k live? e.g., for P=6 (even)



Where do the Ω_k live? e.g., for P=3 (odd)



Consequence for Periodic Input to LTI System

$$x[n] = \sum_{k=\langle P \rangle} A_k e^{j\Omega_k n} \longrightarrow \boxed{H(\Omega)} \longrightarrow y[n] = \sum_{k=\langle P \rangle} H(\Omega_k) A_k e^{j\Omega_k n}$$

We write $\Omega_k = k\Omega_1 = k(2\pi/P)$, to further simplify the notation; so $\Omega_{-k} = -\Omega_k$.

i.e., the frequency response tells us how the system will affect the spectral components in the periodic input. We know the output is periodic, and must have its own Fourier series, with coefficients B_k. So evidently

$$B_k = H(\Omega_k) A_k$$

are the spectral coefficients for y[n]. If we use the alternate normalization, $X_k = A_k P$ and $Y_k = B_k P$, then similarly

$$Y_k = H(\Omega_k) X_k$$

Determining the Fourier Series Coefficients

$$x[n] = \sum_{k=\langle P \rangle} A_k e^{j\Omega_k n}$$ Synthesis equation

$$X_k = A_k P = \sum_{n=\langle P \rangle} x[n] e^{-j\Omega_k n}$$ Analysis equation

$$\Omega_k = k\frac{2\pi}{P}$$

Derivation of equation for A_k

Start with:

$$x[n] = \sum_{m=\langle P \rangle} A_m e^{j\Omega_m n}$$

Multiply both sides by $e^{-j\Omega_k n}$ and sum over P terms:

$$\sum_{n=\langle P \rangle} x[n] e^{-j\Omega_k n} = \sum_{n=\langle P \rangle} \sum_{m=\langle P \rangle} A_m e^{j\Omega_m n} e^{-j\Omega_k n}$$

$$= \sum_{m=\langle P \rangle} A_m \sum_{n=\langle P \rangle} e^{j(m-k)\Omega_1 n}$$

= 0 if m-k ≠ 0, and
= P if m=k

$$= A_k P$$

$$A_k = \frac{1}{P} \sum_{n=\langle P \rangle} x[n] e^{-j\Omega_k n}$$

$$X_k = \sum_{n=\langle P \rangle} x[n] e^{-j\Omega_k n}$$

DTFS Properties

$$x[n] = \sum_{k=\langle P \rangle} A_k e^{j\Omega_k n} \qquad A_k = \frac{1}{P} \sum_{n=\langle P \rangle} x[n] e^{-j\Omega_k n}$$

- x[n] and A_k are both periodic with period P
- If x[n] is real, $A_{-k} = A_k^*$ (i.e., they are complex conjugates)
- A_0 is the average of the x[n] over one period
- $A_{P/2}$ (for even P) is the average of $(-1)^n x[n]$ over one period
- It takes P numbers to specify this periodic x[n], and it takes P numbers to specify its Fourier series coefficients

$$x[n] = \sin(r\frac{2\pi}{P}n)$$

Let's do it "by inspection". First rewrite x[n]:

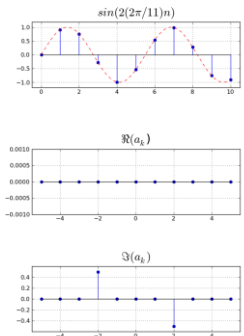
$$x[n] = \frac{1}{2j}e^{jr\frac{2\pi}{P}n} - \frac{1}{2j}e^{j(-r)\frac{2\pi}{P}n}$$

Now x[n] is a sum of complex exponentials and we can determine the A_k directly from the equation:

$$A_r = \frac{1}{2j} = -\frac{j}{2}$$

$$A_{-r} = -\frac{1}{2j} = \frac{j}{2}$$

$$A_k = 0 \quad \text{otherwise}$$



P is *odd* here (=11), so the end points of the frequency scale are at ±(π– (π/P)), not ±π.

$$x[n] = 1 + 2\cos(3\frac{2\pi}{11}n) - 3\sin(5\frac{2\pi}{11}n)$$

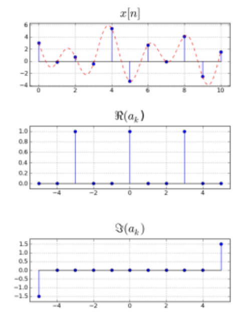
Again, by inspection: since the cos and sin are at different frequencies, we can analyze them separately.

A_0 = average value = 1

$A_{\pm 3}$ = 2(1/2) = 1 [from cos term]

A_{-5} = –3(j/2) = –1.5j [from sin term]
A_5 = –3(–j/2) = 1.5j

A_k = 0 otherwise



Again, P is *odd* here (=11), so the end points of the frequency scale are at ±(π– (π/P)), not ±π.

The DTFS is also good for finite-duration signals!

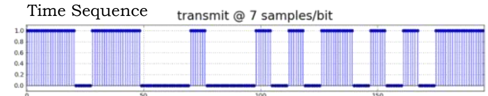
Claim: Over any contiguous interval of length P that we may be interested in --- say n=0,1,…,P–1 for concreteness --- an arbitrary DT signal x[n] can be written in the form

$$x[n] = \sum_{k=\langle P \rangle} A_k e^{j\Omega_k n}$$

What's going on here? If we know we will only be interested in the interval [0,P–1], then it doesn't matter that our representation above will create periodically repeating extensions outside the interval of interest.

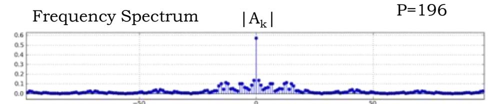
Spectrum of Digital Transmissions

Time Sequence transmit @ 7 samples/bit



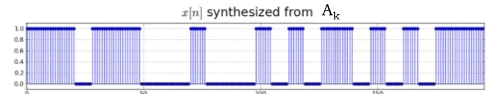
n

Frequency Spectrum $|A_k|$ P=196

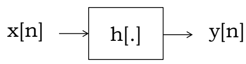


k

$x[n]$ synthesized from A_k



Application



Suppose x[n] is nonzero only over the time interval $[0 , n_x]$, and h[n] is nonzero only over the time interval $[0 , n_h]$.

In what time interval can the non-zero values of y[n] be guaranteed to lie? The interval $[0 , n_x + n_h]$.

Since all the action we are interested in is confined to this interval, choose $P - 1 \geq n_x + n_h$, then use the DTFS to represent x[n] and y[n] over this interval.

This is actually the much more common use of the DTFS!

The Need for Speed: Fast Fourier Transform (FFT)

$$x[n] = \frac{1}{P} \sum_{k=\langle P \rangle} X_k e^{j\Omega_k n} \ , \quad X_k = \sum_{n=\langle P \rangle} x[n] \, e^{-j\Omega_k n}$$

Computing these series involves $O(P^2)$ operations – when P gets large, the computations get very s l o w....

Happily, in 1965 Cooley and Tukey published a fast method for computing the Fourier transform (aka **FFT**, IFFT), rediscovering a technique known to Gauss. This method takes O(P log P) operations.

$P = 1000, \ P^2 = 1000000, \ P \log P \approx 10000$