6.02 Spring 2012 Lecture #14

- Viewing signals in frequency domain
- Discrete-time Fourier Series

U.S. Spectrum Allocation Map

Multiple Transmitters: Frequency Division Multiplexing (FDM)

Choose bandwidths and $k_n$'s so as to avoid overlap! Once signals combine at a given frequency, can't be undone...

Channel "performs addition" by superposing signals ("voltages") from different frequency bands.
Continuous and Discrete-time signals

Example 500Hz carrier tone

\[ \sin\left(\frac{2\pi}{T_p} t\right) \quad \text{[Continuous-time sinewave]} \]

\[ T_p = 2\text{ms} \]

\[ \Omega_0 = \frac{2\pi}{T_p} = \frac{2\pi}{2} = \pi \quad \text{(Discrete-time sinewave)} \]

\[ P = \frac{T_p}{T_s} = 8 \text{ samples per period} \]

Spectrum of Digital Transmissions

Time Sequence

Transmit @ 7 samples/bit

Frequency Spectrum

\[ |A_k| \]

\[ n \]

\[ k \]

\[ \sin(\frac{\omega}{n}) \]

\[ P = \frac{T_p}{T_s} = 8 \text{ samples per period} \]
Representing Periodic Signals in Frequency Domain

Any strictly periodic DT signal of period $P$

$x[n+P]=x[n]$ for all $n$

e.g., $6\sin(2\pi n/P)+0.17 + 4\cos(3(2\pi n/P)+0.82)$

can be written as a weighted combination (generally with complex weights) of $P$ complex exponentials

whose frequencies are consecutive integer multiples of the fundamental frequency $2\pi/P=\Omega_1$

so each exponential term has period $P$

This is the Discrete-Time Fourier Series (DTFS) or discrete spectral representation.

Discrete-Time Fourier Series (DTFS)

If $x[n]$ is periodic with period $P$ (convenient to assume $P$ is even), so $P/2$ is integer, but odd $P$ can be handled too, it can be expressed as the sum of $P$ "spectral components" --- scaled complex exponentials of period $P$:

$$x[n] = \sum_{k=-P/2}^{P/2-1} A_k e^{j\Omega_k n}$$

$k$ ranges over any $P$ consecutive integers. Common choices:

- $k$ for $0$ to $P-1$; $0 \leq k \leq 2\pi/\Omega_1$
- $k$ for $-(P/2)$ to $(P/2)-1$ for even $P$; $-\pi \leq k \leq \pi$ (or $-\pi \leq k \leq \pi$)
- $k$ symmetrically out from $0$ for odd $P$; $-\pi+\pi/2 \leq k \leq \pi-\pi/2$

With the notation $A_k=X_k/P$, we get an alternate (and often used) normalization.

Where do the $\Omega_k$ live?

e.g., for $P=6$ (even)

\[\begin{align*}
\Omega_0 & = 0 \\
\Omega_1 & = 2\pi/6 \\
\Omega_2 & = 4\pi/6 \\
\Omega_3 & = 6\pi/6 \\
\Omega_4 & = 8\pi/6 \\
\Omega_5 & = 10\pi/6
\end{align*}\]

Where do the $\Omega_k$ live?

e.g., for $P=3$ (odd)

\[\begin{align*}
\Omega_0 & = 0 \\
\Omega_1 & = 2\pi/3 \\
\Omega_2 & = 4\pi/3 \\
\Omega_3 & = 6\pi/3 \\
\Omega_4 & = 8\pi/3 \\
\Omega_5 & = 10\pi/3
\end{align*}\]
Consequence for Periodic Input to LTI System

\[ x[n] = \sum_{k=-\infty}^{\infty} A_k e^{j\omega_k n} \]  
\[ y[n] = \sum_{k=-\infty}^{\infty} H(\omega_k) A_k e^{j\omega_k n} \]

We write \( \Omega_0 = \omega_0 - k(2\pi/P) \), to further simplify the notation; so \( \Omega_0 = -\Omega_k \).

i.e., the frequency response tells us how the system will affect the spectral components in the periodic input. We know the output is periodic, and must have its own Fourier series, with coefficients \( B_k \). So evidently

\[ B_k = H(\omega_k) A_k \]

are the spectral coefficients for \( y[n] \). If we use the alternate normalization, \( X_k = A_k P \) and \( Y_k = B_k P \), then similarly

\[ Y_k = H(\omega_k) X_k \]

Determining the Fourier Series Coefficients

\[ x[n] = \sum_{k=-P/2}^{P/2} A_k e^{j\Omega_k n} \]  \hspace{1cm} \text{Synthesis equation}

\[ X_k = A_k P = \sum_{n=-P/2}^{P/2} x[n] e^{-j\Omega_k n} \]  \hspace{1cm} \text{Analysis equation}

\[ \Omega_k = k \frac{2\pi}{P} \]

DTFS Properties

\[ x[n] = \sum_{k=-P/2}^{P/2} A_k e^{j\Omega_k n} \]

\[ A_k = \frac{1}{P} \sum_{n=-P/2}^{P/2} x[n] e^{-j\Omega_k n} \]

\[ X_k = \sum_{n=-P/2}^{P/2} x[n] e^{-j\Omega_k n} \]

- \( x[n] \) and \( A_k \) are both periodic with period \( P \)
- If \( x[n] \) is real, \( A_{-k} = A_k^* \) (i.e., they are complex conjugates)
- \( A_0 \) is the average of the \( x[n] \) over one period
- \( A_{P/2} \) (for even \( P \)) is the average of \( (-1)^n x[n] \) over one period
- It takes \( P \) numbers to specify this periodic \( x[n] \), and it takes \( P \) numbers to specify its Fourier series coefficients
Let’s do it “by inspection”. First rewrite \( x[n] \):
\[
x[n] = \frac{1}{2j} e^{\frac{2\pi}{P} n} - \frac{1}{2j} e^{(\frac{3\pi}{P})n}.
\]
Now \( x[n] \) is a sum of complex exponentials and we can determine the \( A_k \) directly from the equation:
\[
A_k = \frac{1}{2j} e^{\frac{2\pi}{P} n} - \frac{1}{2j} e^{(\frac{3\pi}{P})n}.
\]
\( P \) is odd here (=11), so the end points of the frequency scale are at \( \pm (\pi - \frac{\pi}{P}) \), not \( \pm \pi \).
\[
A_0 = \text{average value } = 1
\]
\[
A_{\pm 3} = 2(1/2) = 1 \quad \text{[from cos term]}
\]
\[
A_{-5} = -(1/2) = -1.5j \quad \text{[from sin term]}
\]
\[
A_5 = 1.5j \quad \text{[from sin term]}
\]
\( A_k = 0 \) otherwise

The DTFS is also good for finite-duration signals!

**Claim**: Over any contiguous interval of length \( P \) that we may be interested in --- say \( n=0,1,\ldots,P-1 \) for concreteness --- an arbitrary DT signal \( x[n] \) can be written in the form
\[
x[n] = \sum_{k=-P}^{P} A_k e^{j\omega_k n}
\]
What’s going on here? If we know we will only be interested in the interval \([0,P-1]\), then it doesn’t matter that our representation above will create periodically repeating extensions outside the interval of interest.
Application

\[ x[n] \rightarrow h[n] \rightarrow y[n] \]

Suppose \( x[n] \) is nonzero only over the time interval \([0, n_x]\) and \( h[n] \) is nonzero only over the time interval \([0, n_h]\).

In what time interval can the non-zero values of \( y[n] \) be guaranteed to lie? The interval \([0, n_x + n_h]\).

Since all the action we are interested in is confined to this interval, choose \( P - 1 \geq n_x + n_h \), then use the DTFS to represent \( x[n] \) and \( y[n] \) over this interval.

This is actually the much more common use of the DTFS!

The Need for Speed:
Fast Fourier Transform (FFT)

\[ x[n] = \sum_{k=-P}^{P-1} X_k e^{j2\pi kn/P}, \quad X_k = \sum_{n=-P}^{P-1} x[n] e^{-j2\pi kn/P} \]

Computing these series involves \( O(P^2) \) operations — when \( P \) gets large, the computations get very slow.

Happily, in 1965 Cooley and Tukey published a fast method for computing the Fourier transform (aka FFT, IFFT), rediscovering a technique known to Gauss. This method takes \( O(P \log P) \) operations.

\( P = 1000, P^2 = 1000000, P \log P = 10000 \)