Massachusetts Institute of Technology
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6.02 **Solutions to Chapter 13** Updated: April 22, 2012

Please send information about errors or omissions to hari; questions are best asked on piazza.

1. In PS 5; please see those solutions.
2. (a)

$$\begin{aligned} |H(\Omega)| &= 1 \quad \text{for } 0 \leq |\Omega| < \Omega_m \ , \\ &= 0 \quad \text{for } \Omega_m \leq |\Omega| \leq \pi \ . \end{aligned}$$

and

$$\angle H(\Omega) = -3\Omega \quad \text{for } 0 \leq |\Omega| < \Omega_m$$

$\angle H(\Omega)$ is a *linear* function of Ω, with slope -3. (The phase is undefined and unimportant for those Ω at which the magnitude is 0.)

(b) A frequency response of $H(\Omega) = e^{-j3\Omega}$ for *all* Ω in $[-\pi, \pi]$ is associated with a unit sample response of $h[n] = \delta[n-3]$, and corresponds to an LTI system that delays its input by 3 samples.

Hence, in terms of the input $x[.]$, the output $y[n] = x[n-3]$.

(c) $A_k = \frac{1}{12}\Big(1 + 2\cos(\Omega_k)\Big)$.

From the expression above, taking $\langle n \rangle$ to be the range $[-6, 5]$ and noting that the only nonzero values of $x[n]$ in this range are $x[-1] = x[0] = x[1] = 1$, we get

$$\begin{aligned} A_k &= \frac{1}{12}\Big(x[-1]e^{j\Omega_k} + x[0]e^{-j0} + x[1]e^{-j\Omega_k}\Big) \\ &= \frac{1}{12}\Big(1 + 2\cos(\Omega_k)\Big) \ . \end{aligned}$$

Equivalently, we could for example take $\langle n \rangle$ to be the range $[0, 11]$. Since the only nonzero values of $x[n]$ in this range are $x[0] = x[1] = x[11] = 1$, we get

$$\begin{aligned} A_k &= \frac{1}{12}\Big(x[0]e^{-j0} + x[1]e^{-j\Omega_k} + x[11]e^{-j11\Omega_k}\Big) \\ &= \frac{1}{12}\Big(x[0]e^{-j0} + x[1]e^{-j\Omega_k} + x[11]e^{j\Omega_k}\Big) = \frac{1}{12}\Big(1 + 2\cos(\Omega_k)\Big) \ , \end{aligned}$$

exactly as before. In arriving at the second equation above, we have used the given equality $\exp(-j11\Omega_1) = \exp(j\Omega_1)$ to conclude (on taking the kth power of each side) that $\exp(-j11k\Omega_1) = \exp(jk\Omega_1)$, i.e., $\exp(-j11\Omega_k) = \exp(j\Omega_k)$.

Check on A_0: We know A_0 should be the *average of the values over one period*, namely $3/12$ or $1/4$, and this is consistent with the above formula, since $\cos(\Omega_0) = \cos(0) = 1$.

Check on A_{-6}: Similiarly, $A_{-6} = A_6$ should be the *"alternating average" of the values over one period*, namely $\frac{1}{12}\sum_{\langle n \rangle}(-1)^n x[n]$, which in this case is $-1/12$. This is again consistent with the above formula, since $\cos(\Omega_{-6}) = \cos(\Omega_6) = \cos(\pi) = -1$.

3. (a) On the real graph: 1 at $\Omega = -\pi$, 0.5 at $\Omega = -2\pi/3$, 1 at $\Omega = -\pi/3$.
   On the imaginary graph: -1 at $\Omega = -\pi/2$. This is based on the observation that for a real-valued signal, the Fourier coefficients must be symmetric, i.e. even for the real part, odd for the imaginary part.

   (b) The period, T, is 12. All frequencies that show up with spectral coefficients must be of the form $\frac{2\pi k}{N}$ for integer k and a constant N. Looking at the real part of the fourier coefficients, we can deduce that $k = 0, 1, 2, 3$ and $N = 3$ can create the frequencies with non-zero components. However, looking at the imaginary part of the spectral coefficients, we observe that $\frac{\pi}{2}$ must also be a multiple of the fundamental frequency. We note that $N = 6$ can satisfy this set of frequency, with $k = 0, 1, 2, 3, 4, 5, 6$. However, the graph only shows $k < N/2$, so N must be equal to 12. This is the period of the signal, and the smallest integer for which $x[n + T] = x[n]$.

   Alternatively: The period $T = 12$ because the least common multiple of $6 = 2\pi/\frac{\pi}{3}$, $4 = 2\pi/\frac{\pi}{2}$, $3 = 2\pi/\frac{2\pi}{3}$ and $2 = 2\pi/\pi$ is 12.

   (c) When a complex exponential $Ae^{j\Omega n}$ passes through an LTI filter, it gets multiplied by $H(e^{j\Omega})$, where $H(\cdot)$ is the frequency response of the filter. Since "delay by 6 samples" is an LTI filter with frequency response $H(e^{j\Omega}) = e^{-j6\Omega}$, the Fourier coefficients at $\Omega = 0$, $\Omega = \pi/3$, $\Omega = \pi/2$, $\Omega = 2\pi/3$, and $\Omega = \pi$ will be multiplied by $e^{-j6\cdot 0} = 1$, $e^{-j6\cdot\pi/3} = 1$, $e^{-j6\cdot\pi/2} = -1$, $e^{-j6\cdot 2\pi/3} = 1$, and $e^{-j6\cdot\pi} = 1$ respectively, the resulting Fourier coefficients will be:

   - On the real graph: 1 at $\Omega = 0$, $\Omega = \pi/3$, $\Omega = 2\pi/3$, and $\Omega = \pi$.
   - On the imaginary graph: -1 at $\Omega = \pi/2$.