LECTURE 5

• Readings: Sections 2.1-2.3, start 2.4

Lecture outline

• Random variables
• Probability mass function (PMF)
• Expectation
• Variance

Random variables

• An assignment of a value (number) to every possible outcome
• Mathematically: A function from the sample space $\Omega$ to the real numbers
  – discrete or continuous values
• Can have several random variables defined on the same sample space

• Notation:
  – random variable $X$
  – numerical value $x$

Probability mass function (PMF)

• (“probability law”, “probability distribution” of $X$)

• Notation:
  \[ p_X(x) = P(X = x) = P(\{\omega \in \Omega \text{ s.t. } X(\omega) = x\}) \]

• $p_X(x) \geq 0$ \[ \sum_x p_X(x) = 1 \]

• Example: $X=$ number of coin tosses until first head
  – assume independent tosses, $P(H) = p > 0$
  \[ p_X(k) = P(X = k) = P(TT\cdots TH) = (1-p)^{k-1}p, \quad k = 1, 2, \ldots \]
  – geometric PMF

How to compute a PMF $p_X(x)$

– collect all possible outcomes for which $X$ is equal to $x$
– add their probabilities
– repeat for all $x$

• Example: Two independent rolls of a fair tetrahedral die

  $F$: outcome of first throw
  $S$: outcome of second throw
  $X = \min(F, S)$

\[ F = \begin{array}{c|c|c|c|c}
1 & 2 & 3 & 4 \\
\hline
1 & 2 & 3 & 4 \\
\hline
2 & 3 & 4 & 1 \\
\hline
3 & 4 & 1 & 2 \\
\hline
4 & 1 & 2 & 3 \\
\end{array} \]

$S = \text{Second roll}$

$F = \text{First roll}$

\[ P_X(2) = \]
**Binomial PMF**

- **X**: number of heads in \( n \) independent coin tosses
- **\( P(H) = p \)**
- Let \( n = 4 \)
\[
P_X(2) = P(HHTT) + P(HTHT) + P(HTTH) + P(THTH) + P(TTHH) \\
= 6p^2(1-p)^2 \\
= \binom{4}{2}p^2(1-p)^2
\]

In general:
\[
p_X(k) = \binom{n}{k}p^k(1-p)^{n-k}, \quad k = 0, 1, \ldots, n
\]

**Expectation**

- **Definition:**
  \[
  E[X] = \sum_x xP_X(x)
  \]
- **Interpretations:**
  - Center of gravity of PMF
  - Average in large number of repetitions of the experiment (to be substantiated later in this course)
- **Example:** Uniform on \( 0, 1, \ldots, n \)

\[
E[X] = 0 \times \frac{1}{n+1} + 1 \times \frac{1}{n+1} + \cdots + n \times \frac{1}{n+1} = \frac{1}{n+1} \sum_{x=0}^{n} x
\]

**Properties of expectations**

- Let \( X \) be a r.v. and let \( Y = g(X) \)
  - Hard: \( E[Y] = \sum_y yP_Y(y) \)
  - Easy: \( E[Y] = \sum_x g(x)p_X(x) \)
- **Caution:** In general, \( E[g(X)] \neq g(E[X]) \)

**Properties:** If \( \alpha, \beta \) are constants, then:

- \( E[\alpha] = \alpha \)
- \( E[\alpha X] = \alpha E[X] \)
- \( E[\alpha X + \beta] = \alpha E[X] + \beta \)

**Variance**

Recall: \( E[g(X)] = \sum_x g(x)p_X(x) \)

- **Second moment:** \( E[X^2] = \sum_x x^2p_X(x) \)
- **Variance**
  \[
  \text{var}(X) = E[(X - E[X])^2] \\
  = \sum_x (x - E[X])^2 p_X(x) \\
  = E[X^2] - (E[X])^2
  \]

**Properties:**

- \( \text{var}(X) \geq 0 \)
- \( \text{var}(\alpha X + \beta) = \alpha^2 \text{var}(X) \)