Lecture 7

- **Readings:** Finish Chapter 2

Lecture outline

- Multiple random variables
  - Joint PMF
  - Conditioning
  - Independence
- More on expectations
- Binomial distribution revisited
- A hat problem

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Review

\[ p_X(x) = P(X = x) \]
\[ p_{X,Y}(x, y) = P(X = x, Y = y) \]
\[ p_{X|Y}(x | y) = P(X = x | Y = y) \]
\[ p_X(x) = \sum_y p_{X,Y}(x, y) \]
\[ p_{X,Y}(x, y) = p_X(x)p_{Y|X}(y | x) \]

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Independent random variables

\[ p_{X,Y,Z}(x, y, z) = p_X(x)p_{Y|X}(y | x)p_{Z|X,Y}(z | x, y) \]

- Random variables \( X, Y, Z \) are independent if:
  \[ p_{X,Y,Z}(x, y, z) = p_X(x) \cdot p_Y(y) \cdot p_Z(z) \]
  for all \( x, y, z \)

- \( p_{X,Y,Z}(x, y, z) \) is 1/20 for all \( x, y, z \)

- Independent?
- What if we condition on \( X \leq 2 \) and \( Y \geq 3 \)?

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Expectations

\[ E[X] = \sum_x xp_X(x) \]
\[ E[g(X, Y)] = \sum_x \sum_y g(x, y)p_{X,Y}(x, y) \]

- In general: \( E[g(X, Y)] \neq g(E[X], E[Y]) \)
- \( E[\alpha X + \beta] = \alpha E[X] + \beta \)
- If \( X, Y \) are independent:
  - \( E[XY] = E[X]E[Y] \)
  - \( E[g(X)h(Y)] = E[g(X)] \cdot E[h(Y)] \)
Variance

- \( \text{Var}(aX) = a^2 \text{Var}(X) \)
- \( \text{Var}(X + a) = \text{Var}(X) \)

Let \( X = a + Y \).

- If \( X, Y \) are independent:
  \( \text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) \)

Examples:
- If \( X = Y \), \( \text{Var}(X + Y) = \)
- If \( X = -Y \), \( \text{Var}(X + Y) = \)
- If \( X, Y \) independent, and \( X = 3Y \), \( \text{Var}(Z) = \)

Binomial mean and variance

- \( X = \) # of successes in \( n \) independent trials
  - probability of success \( p \)
  \( E[X] = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \)

- \( X_i = \begin{cases} 1, & \text{if success in trial } i, \\ 0, & \text{otherwise} \end{cases} \)

- \( E[X_i] = \)
- \( E[X] = \)
- \( \text{Var}(X_i) = \)
- \( \text{Var}(X) = \)

The hat problem

- \( n \) people throw their hats in a box and then pick one at random.
  - \( X \): number of people who get their own hat
  - Find \( E[X] \)

\[
X_i = \begin{cases} 1, & \text{if } i \text{ selects own hat} \\ 0, & \text{otherwise} \end{cases}
\]

- \( X = X_1 + X_2 + \cdots + X_n \)
- \( P(X_i = 1) = \)
- \( E[X_i] = \)
- Are the \( X_i \) independent?
- \( E[X] = \)

Variance in the hat problem

- \( \text{Var}(X) = E[X^2] - (E[X])^2 = E[X^2] - 1 \)

\[
X^2 = \sum_i X_i^2 + \sum_{i,j:i \neq j} X_i X_j
\]

- \( E[X_i^2] = \)

\[
P(X_1 X_2 = 1) = P(X_1 = 1) \cdot P(X_2 = 1 | X_1 = 1)
\]

- \( E[X^2] = \)
- \( \text{Var}(X) = \)