Lecture 12

- **Readings:** Section 4.3; parts of Section 4.5 (mean and variance only; no transforms)

**Lecture outline**

- Conditional expectation
  - Law of iterated expectations
  - Law of total variance
- Sum of a random number of independent r.v.'s
  - mean, variance

<table>
<thead>
<tr>
<th>Conditional expectations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Given the value $y$ of a r.v. $Y$:</td>
</tr>
<tr>
<td>$E[X</td>
</tr>
</tbody>
</table>

(integral in continuous case)

- Stick example: stick of length $\ell$ break at uniformly chosen point $Y$ break again at uniformly chosen point $X$

  $E[X | Y = y] = \frac{y}{2}$ (number)

  $E[X | Y] = \frac{Y}{2}$ (r.v.)

- **Law of iterated expectations:**
  
  $E[E[X | Y]] = \sum_y E[X | Y = y] p_Y(y) = E[X]$

- In stick example:
  
  $E[X] = E[E[X | Y]] = E[Y/2] = \ell/4$

---

**var($X | Y$) and its expectation**

- $\text{var}(X | Y = y) = \mathbb{E}[(X - \mathbb{E}[X | Y = y])^2 | Y = y]$

- $\text{var}(X | Y)$: a r.v. with value $\text{var}(X | Y = y)$ when $Y = y$

- **Law of total variance:**
  
  $\text{var}(X) = \mathbb{E}[\text{var}(X | Y)] + \text{var}(\mathbb{E}[X | Y])$

**Proof:**

(a) Recall: $\text{var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

(b) $\text{var}(X | Y) = \mathbb{E}[X^2 | Y] - (\mathbb{E}[X | Y])^2$

(c) $\mathbb{E}[\text{var}(X | Y)] = \mathbb{E}[X^2] - \mathbb{E}[(\mathbb{E}[X | Y])^2]$

(d) $\text{var}(\mathbb{E}[X | Y]) = \mathbb{E}[(\mathbb{E}[X | Y])^2] - (\mathbb{E}[X])^2$

Sum of right-hand sides of (c), (d):

$\mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \text{var}(X)$

---

**Section means and variances**

Two sections: $y = 1$ (10 students); $y = 2$ (20 students)

$y = 1: \frac{1}{10} \sum_{i=1}^{10} x_i = 90 \quad y = 2: \frac{1}{20} \sum_{i=1}^{20} x_i = 60$

$E[X] = \frac{1}{30} \sum_{i=1}^{30} x_i = \frac{90 \cdot 10 + 60 \cdot 20}{30} = 70$

$E[X | Y = 1] = 90, \quad E[X | Y = 2] = 60$

$E[X | Y] = \begin{cases} 90, & \text{w.p. } 1/3 \\ 60, & \text{w.p. } 2/3 \end{cases}$

$E[\text{var}(X | Y)] = \frac{1}{3} \cdot 90 + \frac{2}{3} \cdot 60 = 70 = E[X]$

$\text{var}(\mathbb{E}[X | Y]) = \frac{1}{3} (90 - 70)^2 + \frac{2}{3} (60 - 70)^2 = \frac{600}{3} = 200$
Section means and variances (ctd.)

\[
\frac{1}{10} \sum_{i=1}^{10} (x_i - 90)^2 = 10 \\
\frac{1}{20} \sum_{i=11}^{30} (x_i - 60)^2 = 20
\]

\[\text{var}(X \mid Y = 1) = 10 \quad \text{var}(X \mid Y = 2) = 20\]

\[\text{var}(X \mid Y) = \begin{cases} 
10, & \text{w.p. } 1/3 \\
20, & \text{w.p. } 2/3
\end{cases}\]

\[E[\text{var}(X \mid Y)] = \frac{1}{3} \cdot 10 + \frac{2}{3} \cdot 20 = \frac{60}{3}\]

\[E[\text{var}(X)] = \text{var}(E[X \mid Y])
\]

\[= \frac{50}{3} + 200
\]

\[= \text{(average variability within sections)}
\]

\[+ \text{(variability between sections)}\]

Example

\[\text{var}(X) = E[\text{var}(X \mid Y)] + \text{var}(E[X \mid Y])\]

\[E[X \mid Y = 1] = \quad E[X \mid Y = 2] = \]

\[\text{var}(X \mid Y = 1) = \quad \text{var}(X \mid Y = 2) = \]

\[E[X] =
\]

\[E[\text{var}(X \mid Y)] =
\]

Sum of a random number of independent r.v.'s

- \(N\): number of stores visited
  (\(N\) is a nonnegative integer r.v.)

- \(X_i\): money spent in store \(i\)
  - \(X_i\) assumed i.i.d.
  - independent of \(N\)

- Let \(Y = X_1 + \cdots + X_N\)

\[E[Y \mid N = n] = E[X_1 + X_2 + \cdots + X_n \mid N = n] = E[X_1] + E[X_2] + \cdots + E[X_n] = n E[X]\]

\[E[Y] = E[E[Y \mid N]] = E[N E[X]] = E[N] E[X]\]

Variance of sum of a random number of independent r.v.'s

- \(\text{var}(Y) = E[\text{var}(Y \mid N)] + \text{var}(E[Y \mid N])\)

- \(E[Y \mid N] = N E[X]\)

\[\text{var}(E[Y \mid N]) = (E[X])^2 \text{var}(N)\]

- \(\text{var}(Y \mid N = n) = n \text{var}(X)\)

\[E[\text{var}(Y \mid N)] = E[N \text{var}(X)]\]

\[\text{var}(Y) = E[\text{var}(Y \mid N)] + \text{var}(E[Y \mid N]) = E[N] \text{var}(X) + (E[X])^2 \text{var}(N)\]