The Bernoulli process

- A sequence of independent Bernoulli trials

  - At each trial, \( i \):
    - \( P(\text{success}) = P(X_i = 1) = p \)
    - \( P(\text{failure}) = P(X_i = 0) = 1 - p \)

- Examples:
  - Sequence of lottery wins/losses
  - Sequence of ups and downs of the Dow Jones
  - Arrivals (each second) to a bank
  - Arrivals (at each time slot) to server

Random processes

- First view:
  sequence of random variables \( X_1, X_2, \ldots \)

- \( E[X_i] = \)
- \( \text{Var}(X_i) = \)

- Second view:
  what is the right sample space?

- \( P(X_i = 1 \text{ for all } i) = \)

- Random processes we will study:
  - Bernoulli process
    (memoryless, discrete time)
  - Poisson process
    (memoryless, continuous time)
  - Markov chains
    (with memory/dependence across time)
Interarrival times

- $T_1$: number of trials until first success
  - $P(T_1 = t) = \ldots$
  - Memoryless property
  - $E[T_1] = \ldots$
  - $Var(T_1) = \ldots$

- If you buy a lottery ticket every day, what is the distribution of the length of the first string of losing days?

Time of the $k$th arrival

- Given that first arrival was at time $t$
  - i.e., $T_1 = t$:
    - additional time, $T_2$, until next arrival
    - has the same (geometric) distribution
    - independent of $T_1$

- $Y_k$: number of trials to $k$th success
  - $E[Y_k] = \ldots$
  - $Var(Y_k) = \ldots$
  - $P(Y_k = t) = \ldots$

Splitting of a Bernoulli Process

(using independent coin flips)

Yields Bernoulli processes

Merging of Indep. Bernoulli Processes

Yields a Bernoulli process
(collisions are counted as one arrival)

Poisson approximation to binomial

- Number of arrivals in $n$ slots is binomial
  $p_S(k) = \frac{n!}{(n-k)k!} p^k (1-p)^{n-k}, \quad$ for $k \geq 0$

- Interesting to think of:
  $n \to \infty$ with $\lambda = np$ constant
  $p_S(k) = \frac{n!}{(n-k)k!} p^k (1-p)^{n-k}$
  $= \frac{n(n-1) \cdots (n-k+1)}{k!} \cdot \frac{\lambda^k}{n^k} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k}$
  $= \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-k+1}{n} \cdot \frac{\lambda^k}{k!} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k}$

- For any fixed $k \geq 0$,
  $\lim_{n \to \infty} (1 - \lambda/n)^{n-k} = e^{-\lambda}, \quad$ so:
  $\lim_{n \to \infty} p_S(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 1, 2, \ldots$