LECTURE 14
The Poisson process

- **Readings**: Start Section 6.2.

Lecture outline
- Review of Bernoulli process
- Definition of Poisson process
- Distribution of number of arrivals
- Distribution of interarrival times
- Other properties of the Poisson process

bernoulli review
- Discrete time; success probability \( p \)
- Number of arrivals in \( n \) time slots: binomial pmf
- Interarrival times: geometric pmf
- Time to \( k \) arrivals: Pascal pmf
- Memorylessness

Definition of the Poisson process

- Time homogeneity:
  \[ P(k, \tau) = \text{Prob. of } k \text{ arrivals in interval of duration } \tau \]
- Numbers of arrivals in disjoint time intervals are **independent**
- Small interval probabilities:
  For VERY small \( \delta \):
  \[
  P(k, \delta) \approx \begin{cases} 
    1 - \lambda \delta, & \text{if } k = 0; \\
    \lambda \delta, & \text{if } k = 1; \\
    0, & \text{if } k > 1.
  \end{cases}
  \]
  - \( \lambda \): “arrival rate”

PMF of Number of Arrivals \( N \)

- Finely discretize \([0, t] \): approximately Bernoulli
- \( N_t \) (of discrete approximation): binomial
- Taking \( \delta \to 0 \) (or \( n \to \infty \)) gives:
  \[
  P(k, \tau) = \frac{(\lambda \tau)^k e^{-\lambda \tau}}{k!}, \quad k = 0, 1, \ldots
  \]
- \( E[N_t] = \lambda t \), \( \text{var}(N_t) = \lambda t \)
**Example**

- You get email according to a Poisson process at a rate of $\lambda = 5$ messages per hour. You check your email every thirty minutes.

- $\text{Prob(} \text{no new messages}) =$

- $\text{Prob(} \text{one new message}) =$

**Interarrival Times**

- $Y_k$ time of $k$th arrival

- **Erlang** distribution:  
  $$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \quad y \geq 0$$

  - Time of first arrival ($k = 1$): 
    - exponential:  
      $$f_{Y_1}(y) = \lambda e^{-\lambda y}, \quad y \geq 0$$
      - Memoryless property: The time to the next arrival is independent of the past

**Bernoulli/Poisson Relation**

- Sum of independent Poisson **random variables** is Poisson

- Sum of independent Poisson **processes** is Poisson

- What is the probability that the next arrival comes from the first process?

**Adding Poisson Processes**

<table>
<thead>
<tr>
<th>POISSON</th>
<th>BERNOULLI</th>
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</thead>
<tbody>
<tr>
<td>Times of Arrival</td>
<td>Continuous</td>
</tr>
<tr>
<td>Arrival Rate</td>
<td>$\lambda$/unit time</td>
</tr>
<tr>
<td>PMF of # of Arrivals</td>
<td>Poisson</td>
</tr>
<tr>
<td>Interarrival Time Distr.</td>
<td>Exponential</td>
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<tr>
<td>Time to $k$-th arrival</td>
<td>Erlang</td>
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