LECTURE 17

Markov Processes – II

• Readings: Section 7.3

Lecture outline

• Review
  • Steady-State behavior
    – Steady-state convergence theorem
    – Balance equations
  • Birth-death processes

Review

• Discrete state, discrete time, time-homogeneous
  – Transition probabilities $p_{ij}$
  – Markov property

• $r_{ij}(n) = P(X_n = j \mid X_0 = i)$

• Key recursion:
  
  $r_{ij}(n) = \sum_k r_{ik}(n-1)p_{kj}$

Warmup

$P(X_1 = 2, X_2 = 6, X_3 = 7 \mid X_0 = 1) =$

$P(X_4 = 7 \mid X_0 = 2) =$

Recurrent and transient states

• State $i$ is recurrent if:
  starting from $i$,
  and from wherever you can go,
  there is a way of returning to $i$

• If not recurrent, called transient

• Recurrent class:
  collection of recurrent states that
  “communicate” to each other
  and to no other state

Periodic states

• The states in a recurrent class are periodic if they can be grouped into $d > 1$ groups so that all transitions from one group lead to the next group
Steady-State Probabilities

- Do the \( r_{ij}(n) \) converge to some \( \pi_j \)? (independent of the initial state \( i \))
- Yes, if:
  - recurrent states are all in a single class, and
  - single recurrent class is not periodic
- Assuming “yes,” start from key recursion
  \[ r_{ij}(n) = \sum_k r_{ik}(n-1)p_{kj} \]
  - take the limit as \( n \to \infty \)
  \[ \pi_j = \sum_k \pi_k p_{kj}, \quad \text{for all } j \]
  - Additional equation:
    \[ \sum_j \pi_j = 1 \]

Visit frequency interpretation

\[ \pi_j = \sum_k \pi_k p_{kj} \]

- (Long run) frequency of being in \( j \): \( \pi_j \)
- Frequency of transitions \( k \to j \): \( \pi_k p_{kj} \)
- Frequency of transitions into \( j \): \( \sum_k \pi_k p_{kj} \)

Visit frequency interpretation

\[ \pi_j = \sum_k \pi_k p_{kj} \]

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Example

Birth-death processes

- Special case: \( p_i = p \) and \( q_i = q \) for all \( i \)
  \[ \rho = p/q = \text{load factor} \]
  \[ \pi_{i+1} = \pi_i \frac{p}{q} = \pi_i \rho \]
  \[ \pi_i = \pi_0 \rho^i, \quad i = 0, 1, \ldots, m \]
- Assume \( p < q \) and \( m \approx \infty \)
  \[ \pi_0 = 1 - \rho \]
  \[ E[X_n] = \frac{\rho}{1 - \rho} \quad \text{(in steady-state)} \]