1. Let us draw the region where $f_{X,Y}(x,y)$ is nonzero:

The joint PDF has to integrate to 1. From $\int_{x=1}^{x=2} \int_{y=0}^{y=x} ax \, dy \, dx = \frac{7}{3}a = 1$, we get $a = \frac{3}{7}$.

(b) \[ f_Y(y) = \int f_{X,Y}(x,y) \, dy = \begin{cases} \int_{y=1}^{y=x} \frac{3}{7} x \, dx, & \text{if } 0 \leq y \leq 1, \\ \int_{y=x}^{y=2} \frac{3}{7} x \, dx, & \text{if } 1 < y \leq 2, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{9}{14}, & \text{if } 0 \leq y \leq 1, \\ \frac{3}{14}(4-y^2), & \text{if } 1 < y \leq 2, \\ 0, & \text{otherwise}. \end{cases} \]

(c) \[ f_{X|Y}(x | \frac{3}{2}) = \frac{f_{X,Y}(x, \frac{3}{2})}{f_Y(\frac{3}{2})} = \frac{8}{7} x, \quad \text{for } \frac{3}{2} \leq x \leq 2 \text{ and 0 otherwise.} \]

Then, \[ E \left[ X \mid Y = \frac{3}{2} \right] = \int_{3/2}^{2} \frac{1}{7} x \, dx = \frac{4}{7}. \]

(d) We use the technique of first finding the CDF and differentiating it to get the PDF.

\[
F_Z(z) = P(Z \leq z) = P(Y - X \leq z)
\]

\[
= \begin{cases} 0, & \text{if } z < -2, \\ \int_{x=-z}^{x=2} \int_{y=0}^{y=x+z} \frac{3}{7} x \, dy \, dx = \frac{8}{7} + \frac{6}{7}z - \frac{1}{14}z^3, & \text{if } -2 \leq z \leq -1, \\ \int_{x=1}^{x=2} \int_{y=x-z}^{y=2} \frac{3}{7} x \, dy \, dx = 1 + \frac{9}{14}z, & \text{if } -1 < z \leq 0, \\ 1, & \text{if } 0 < z. \end{cases}
\]

\[
f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} \frac{6}{7} - \frac{3}{14}z^2, & \text{if } -2 \leq z \leq -1, \\ \frac{3}{14}, & \text{if } -1 < z \leq 0, \\ 0, & \text{otherwise}. \end{cases}
\]
2. The PDF of $Z$, $f_Z(z)$, can be readily computed using the convolution integral:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(t)f_Y(z-t) \, dt.$$ 

For $z \in [-1, 0]$,

$$f_Z(z) = \int_{-1}^{z} \frac{3}{4} \frac{3}{4} (1-t^2) \, dt = \frac{1}{4} \left( z - \frac{z^3}{3} + \frac{2}{3} \right).$$

For $z \in [0, 1]$,

$$f_Z(z) = \int_{-1}^{z} \frac{3}{4} \frac{3}{4} (1-t^2) \, dt = \frac{1}{4} \left( 1 - \frac{z^3}{3} + \frac{(z-1)^3}{3} \right).$$

For $z \in [1, 2]$,

$$f_Z(z) = \int_{z-1}^{1} \frac{3}{4} \frac{3}{4} (1-t^2) \, dt + \int_{-1}^{z-2} \frac{3}{4} \frac{3}{4} (1-t^2) \, dt = \frac{1}{4} \left( z + \frac{(z-1)^3}{3} - \frac{2(z-2)^3}{3} - 1 \right).$$

For $z \in [2, 3]$,

$$f_Z(z) = \int_{z-3}^{z-2} \frac{2}{3} \frac{3}{4} (1-t^2) \, dt = \frac{1}{6} (3 + (z-3)^3 - (z-2)^3).$$

For $z \in [3, 4]$,

$$f_Z(z) = \int_{z-3}^{1} \frac{2}{3} \frac{3}{4} (1-t^2) \, dt = \frac{1}{6} (11 - 3z + (z-3)^3).$$

A sketch of $f_Z(z)$ is provided below.

\[ \text{A sketch of } f_Z(z) \]

3. (a) $X_1$ and $X_2$ are negatively correlated. Intuitively, a large number of tosses that result in a 1 suggests a smaller number of tosses that result in a 2.

(b) Let $A_t$ (respectively, $B_t$) be a Bernoulli random variable that is equal to 1 if and only if the $t$th toss resulted in 1 (respectively, 2). We have $\mathbf{E}[A_tB_t] = 0$ (since $A_t \neq 0$ implies $B_t = 0$) and

$$\mathbf{E}[A_tB_s] = \mathbf{E}[A_t]\mathbf{E}[B_s] = \frac{1}{k} \cdot \frac{1}{k} \quad \text{for} \quad s \neq t.$$ 

Thus,

$$\mathbf{E}[X_1X_2] = \mathbf{E}[(A_1 + \cdots + A_n)(B_1 + \cdots + B_n)] = n\mathbf{E}[A_1(B_1 + \cdots + B_n)] = n(n-1) \cdot \frac{1}{k} \cdot \frac{1}{k}$$
\[
\text{cov}(X_1, X_2) = \mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2]
\]
\[
= \frac{n(n-1)}{k^2} - \frac{n^2}{k^2} = -\frac{n}{k^2}.
\]

The covariance of \(X_1\) and \(X_2\) is negative as expected.

4. (a) If \(X\) takes a value \(x\) between \(-1\) and \(1\), the conditional PDF of \(Y\) is uniform between \(-2\) and \(2\). If \(X\) takes a value \(x\) between \(1\) and \(2\), the conditional PDF of \(Y\) is uniform between \(-1\) and \(1\).

Similarly, if \(Y\) takes a value \(y\) between \(-1\) and \(1\), the conditional PDF of \(X\) is uniform between \(-1\) and \(2\). If \(Y\) takes a value \(y\) between \(1\) and \(2\), or between \(-2\) and \(-1\), the conditional PDF of \(X\) is uniform between \(-1\) and \(1\).

(b) We have
\[
\mathbb{E}[X \mid Y = y] = \begin{cases} 0, & \text{if } -2 \leq y \leq -1, \\ 1/2, & \text{if } -1 < y \leq 1, \\ 0, & \text{if } 1 \leq y \leq 2, \end{cases}
\]
and
\[
\text{var}(X \mid Y = y) = \begin{cases} 1/3, & \text{if } -2 \leq y \leq -1, \\ 3/4, & \text{if } -1 < y \leq 1, \\ 1/3, & \text{if } 1 \leq y \leq 2. \end{cases}
\]

It follows that \(\mathbb{E}[X] = 3/10\) and \(\text{var}(X) = 193/300\).

(c) By symmetry, we have \(\mathbb{E}[Y \mid X] = 0\) and \(\mathbb{E}[Y] = 0\). Furthermore, \(\text{var}(Y \mid X = x)\) is the variance of a uniform PDF (whose range depends on \(x\)), and
\[
\text{var}(Y \mid X = x) = \begin{cases} 4/3, & \text{if } -1 \leq x \leq 1, \\ 1/3, & \text{if } 1 < x \leq 2. \end{cases}
\]

Using the law of total variance, we obtain
\[
\text{var}(Y) = \mathbb{E}[^\text{var}(Y \mid X)] = \frac{4}{5} \cdot \frac{4}{3} + \frac{1}{5} \cdot \frac{1}{3} = 17/15.
\]

5. First let us write out the properties of all of our random variables. Let us also define \(K\) to be the number of members attending a meeting and \(B\) to be the Bernoulli random variable describing whether or not a member attends a meeting.

\[
\mathbb{E}[N] = \frac{1}{1-p}, \quad \text{var}(N) = \frac{p}{(1-p)^2},
\]
\[
\mathbb{E}[M] = \frac{1}{\lambda}, \quad \text{var}(M) = \frac{1}{\lambda^2},
\]
\[
\mathbb{E}[B] = q, \quad \text{var}(B) = q(1-q).
\]

(a) Since \(K = B_1 + B_2 + \cdots + B_N\),
\[
\mathbb{E}[K] = \mathbb{E}[N] \cdot \mathbb{E}[B] = \frac{q}{1-p},
\]
\[
\text{var}(K) = \mathbb{E}[N] \cdot \text{var}(B) + (\mathbb{E}[B])^2 \cdot \text{var}(N) = \frac{q(1-q)}{1-p} + \frac{pq^2}{(1-p)^2}.
\]
(b) Let $G$ be the total money brought to the meeting. Then $G = M_1 + M_2 + \cdots + M_K$.

\[
\begin{align*}
E[G] &= E[M] \cdot E[K] = \frac{q}{\lambda(1 - p)}, \\
\text{var}(G) &= \text{var}(M) \cdot E[K] + (E[M])^2 \text{var}(K) \\
&= \frac{q}{\lambda^2(1 - p)} + \frac{1}{\lambda^2} \left( \frac{q(1 - q)}{1 - p} + \frac{pq^2}{(1 - p)^2} \right).
\end{align*}
\]

G1. (a) Let $X_1, X_2, \ldots X_n$ be independent, identically distributed (IID) random variables. We note that

\[
E[X_1 + \cdots + X_n \mid X_1 + \cdots + X_n = x_0] = x_0.
\]

It follows from the linearity of expectations that

\[
x_0 = E[X_1 + \cdots + X_n \mid X_1 + \cdots + X_n = x_0] = E[X_1 \mid X_1 + \cdots + X_n = x_0] + \cdots + E[X_n \mid X_1 + \cdots + X_n = x_0].
\]

Because the $X_i$’s are identically distributed, we have the following relationship.

\[
E[X_i \mid X_1 + \cdots + X_n = x_0] = E[X_j \mid X_1 + \cdots + X_n = x_0], \text{ for any } 1 \leq i \leq n, 1 \leq j \leq n.
\]

Therefore,

\[
nE[X_1 \mid X_1 + \cdots + X_n = x_0] = \frac{x_0}{n}.
\]

(b) Note that we can rewrite $E[X_1 \mid S_n = s_n, S_{n+1} = s_{n+1}, \ldots, S_{2n} = s_{2n}]$ as follows:

\[
\begin{align*}
E[X_1 \mid S_n = s_n, S_{n+1} = s_{n+1}, \ldots, S_{2n} = s_{2n}] &= E[X_1 \mid S_n = s_n, X_{n+1} = s_{n+1} - s_n, X_{n+2} = s_{n+2} - s_{n+1}, \ldots, X_{2n} = s_{2n} - s_{2n-1}] \\
&= E[X_1 \mid S_n = s_n],
\end{align*}
\]

where the last equality holds due to the fact that the $X_i$’s are independent. We also note that

\[
E[X_1 + \cdots + X_n \mid S_n = s_n] = E[S_n \mid S_n = s_n] = s_n.
\]

It follows from the linearity of expectations that

\[
E[X_1 + \cdots + X_n \mid S_n = s_n] = E[X_1 \mid S_n = s_n] + \cdots + E[X_n \mid S_n = s_n].
\]

Because the $X_i$’s are identically distributed, we have the following relationship:

\[
E[X_i \mid S_n = s_n] = E[X_j \mid S_n = s_n], \text{ for any } 1 \leq i \leq n, 1 \leq j \leq n.
\]

Therefore,

\[
E[X_1 + \cdots + X_n \mid S_n = s_n] = nE[X_1 \mid S_n = s_n] = s_n \Rightarrow E[X_1 \mid S_n = s_n] = \frac{s_n}{n}.
\]