Recitation 10 Solutions
(6.041/6.431 Spring 2010 Quiz 1 Solutions)

Question 1

1.1. Which one of the following statements is true?
   
   (a) $P(A \cap B)$ may be larger than $P(A)$.
   
   (b) The variance of $X$ may be larger than the variance of $2X$.
   
   (c) If $A^c \cap B^c = \emptyset$, then $P(A \cup B) = 1$.
   
   (d) If $A^c \cap B^c = \emptyset$, then $P(A \cap B) = P(A)P(B)$.
   
   (e) If $P(A) > 1/2$ and $P(B) > 1/2$, then $P(A \cup B) = 1$.

   **Answer:** (c) is true because $A \cup B = (A^c \cap B^c)^c = \emptyset^c = \Omega$.

1.2. Which one of the following statements is true?
   
   (a) If $E[X] = 0$, then $P(X > 0) = P(X < 0)$.
   
   (b) $P(A) = P(A \mid B) + P(A \mid B^c)$
   
   (c) $P(B \mid A) + P(B \mid A^c) = 1$
   
   (d) $P(B \mid A) + P(B^c \mid A^c) = 1$
   
   (e) $P(B \mid A) + P(B^c \mid A) = 1$

   **Answer:** (e) is true because $B$ and $B^c$ partition $\Omega$.

Question 2

Heather and Taylor play a game using independent tosses of an unfair coin. A head comes up on any toss with probability $p$, where $0 < p < 1$. The coin is tossed repeatedly until either the second time head comes up, in which case Heather wins; or the second time tail comes up, in which case Taylor wins. Note that a full game involves 2 or 3 tosses.

2.1. Consider a probabilistic model for the game in which the outcomes are the sequences of heads and tails in a full game. Provide a list of the outcomes and their probabilities of occurring.

   Because of the independence of the coin tosses, the outcomes and their probabilities are as follows:

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>HH</td>
<td>$p^2$</td>
</tr>
<tr>
<td>HTH</td>
<td>$p^2(1-p)$</td>
</tr>
<tr>
<td>HTT</td>
<td>$p(1-p)^2$</td>
</tr>
<tr>
<td>THH</td>
<td>$p^2(1-p)$</td>
</tr>
<tr>
<td>THT</td>
<td>$p(1-p)^2$</td>
</tr>
<tr>
<td>TT</td>
<td>$(1-p)^2$</td>
</tr>
</tbody>
</table>

2.2. What is the probability that Heather wins the game?

   The event of Heather winning is $\{HH, HTH, THH\}$. Adding the probabilities of the outcomes in this event gives $p^2 + p^2(1-p) + p^2(1-p) = p^2(3-2p)$. 

2.3. What is the conditional probability that Heather wins the game given that head comes up on the first toss?

\[
P(\{\text{Heather wins} \} \mid \{\text{first toss H}\}) = \frac{P(\{\text{Heather wins} \} \cap \{\text{first toss H}\})}{P(\{\text{first toss H}\})} = \frac{P(\{\text{HH, HTH}\})}{P(\{\text{first toss H}\})} = \frac{p^2 + p^2(1-p)}{p} = p(2-p)
\]

2.4. What is the conditional probability that head comes up on the first toss given that Heather wins the game?

\[
P(\{\text{first toss H} \} \mid \{\text{Heather wins} \}) = \frac{P(\{\text{first toss H} \} \cap \{\text{Heather wins} \})}{P(\{\text{Heather wins} \})} = \frac{P(\{\text{HH, HTH}\})}{P(\{\text{Heather wins} \})} = \frac{p^2 + p^2(1-p)}{p^2(3-2p)} = \frac{2-p}{3-2p}
\]

Question 3

A casino game using a fair 4-sided die (with labels 1, 2, 3, and 4) is offered in which a basic game has 1 or 2 die rolls:

- If the first roll is a 1, 2, or 3, the player wins the amount of the die roll, in dollars, and the game is over.
- If the first roll is a 4, the player wins $2 and the amount of a second ("bonus") die roll in dollars.

Let \(X\) be the payoff in dollars of the basic game.

3.1. Find the PMF of \(X\), \(p_X(x)\).

Define a probabilistic model in which the outcomes are the sequences of rolls in a full game. The outcomes, their probabilities, and the resulting values of \(X\) are as follows:

<table>
<thead>
<tr>
<th>(\omega)</th>
<th>(p(\omega))</th>
<th>(X(\omega))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>1/4</td>
<td>1</td>
</tr>
<tr>
<td>(2)</td>
<td>1/4</td>
<td>2</td>
</tr>
<tr>
<td>(3)</td>
<td>1/4</td>
<td>3</td>
</tr>
<tr>
<td>(4, 1)</td>
<td>1/16</td>
<td>3</td>
</tr>
<tr>
<td>(4, 2)</td>
<td>1/16</td>
<td>4</td>
</tr>
<tr>
<td>(4, 3)</td>
<td>1/16</td>
<td>5</td>
</tr>
<tr>
<td>(4, 4)</td>
<td>1/16</td>
<td>6</td>
</tr>
</tbody>
</table>
By gathering the probabilities of the possible values for $X$, we obtain

$$p_X(x) = \begin{cases} 
1/4, & \text{for } x = 1, 2; \\
5/16, & \text{for } x = 3; \\
1/16, & \text{for } x = 4, 5, 6; \\
0, & \text{otherwise.}
\end{cases}$$

3.2. Find $E[X]$.

It does not take too much arithmetic to compute $E[X]$ using the PMF computed in the previous part. A more elegant solution is to use the total expectation theorem. Let $A$ be the event that the first roll is a 4. Then

$$E[X] = P(A) E[X | A] + P(A^c) E[X | A^c] = \frac{21}{8},$$

where $E[X | A] = 4.5$ because the conditional distribution is uniform on $\{3, 4, 5, 6\}$; and $E[X | A^c] = 2$ because the conditional distribution is uniform on $\{1, 2, 3\}$.

3.3. Find the conditional PMF of the result of the first die roll given that $X = 3$. (Use a reasonable notation that you define explicitly.)

Let $Z$ be the result of the first die roll, and let $B = \{X = 3\}$. By definition of conditioning,

$$p_{Z|B}(z) = \frac{P(\{Z = z\} \cap B)}{P(B)}.$$ 

By using values tabulated above,

$$p_{Z|B}(z) = \begin{cases} 
4/5, & \text{for } z = 3; \\
1/5, & \text{for } z = 4; \\
0, & \text{otherwise.}
\end{cases}$$

3.4. Now consider an extended game that can have any number of bonus rolls. Specifically:

- Any roll of a 1, 2, or 3 results in the player winning the amount of the die roll, in dollars, and the termination of the game.
- Any roll of a 4 results in the player winning $2 and continuation of the game.

Let $Y$ denote the payoff in dollars of the extended game. Find $E[Y]$.

One could explicitly find the PMF of $Y$, but this is unnecessarily messy. Instead, let $L$ be the payoff of the last roll and let $W$ be the payoff of all of the earlier rolls. Then $Y = W + L$ by construction, and $E[Y] = E[W] + E[L]$.

The last roll is uniformly distributed on $\{1, 2, 3\}$, so $E[L] = 2$. The winnings on earlier rolls is $2(N - 1)$ where $N$ is the number of rolls in the game. Since termination of the game can be seen as “success” on a Bernoulli trial with success probability of $3/4$, $N$ has the geometric distribution with parameter $3/4$. Thus,

$$E[W] = E[2(N - 1)] = 2E[N] - 2 = 2 \cdot \frac{4}{3} - 2 = \frac{2}{3}.$$ 

Combining the calculations,

$$E[Y] = E[W] + E[L] = \frac{2}{3} + 2 = \frac{8}{3}.$$ 

(Many other methods of solution are possible.)