1. (a) We begin by writing the definition for $E[Z \mid X, Y]$

$$E[Z \mid X = x, Y = y] = \sum_z z p_{Z \mid X,Y}(z \mid x, y)$$

Since $E[Z \mid X, Y]$ is a function of the random variables $X$ and $Y$, and is equal to $E[Z \mid X = x, Y = y]$ whenever $X = x$ and $Y = y$, which happens with probability $p_{X,Y}(x, y)$, using the expected value rule, we have

$$E[E[Z \mid X, Y]] = \sum_x \sum_y E[Z \mid X = x, Y = y] p_{X,Y}(x, y)$$

$$= \sum_x \sum_y \sum_z z p_{Z \mid X,Y}(z \mid x, y) p_{X,Y}(x, y)$$

$$= \sum_x \sum_y \sum_z z p_{X,Y,Z}(x, y, z)$$

$$= E[Z]$$

(b) We start with the definition for $E[Z \mid X, Y]$ which is a function of the random variables $X$ and $Y$, and is equal to $E[Z \mid X = x, Y = y]$ whenever $X = x$ and $Y = y$, so

$$E[Z \mid X = x, Y = y] = \sum_z z p_{Z \mid X,Y}(z \mid x, y)$$

Proceeding as above, but conditioning on the event $X = x$, we have

$$E[E[Z \mid X, Y = y] \mid X = x] = \sum_y E[Z \mid X = x, Y = y] p_{Y \mid X}(y \mid x)$$

$$= \sum_y \sum_z z p_{Z \mid X,Y}(z \mid x, y) p_{Y \mid X}(y \mid x)$$

$$= \sum_y \sum_z z p_{Y,Z \mid X}(y, z \mid x)$$

$$= E[Z \mid X = x]$$

Since this is true for all possible values of $x$, we have $E[E[E[Z \mid Y, X] \mid X] = E[Z \mid X]$.  

(c) We take expectations of both sides of the formula in part (b) to obtain

$$E[E[E[Z \mid X] = E[E[E[Z \mid X, Y] \mid X]].$$

By the law of iterated expectations, the left-hand side above is $E[Z]$, which establishes the desired result.

2. Let $Y$ be the length of the piece after we break for the first time. Let $X$ be the length after we break for the second time.
(a) The law of iterated expectations states:
\[ E[X] = E[E[X | Y]] \]
We have \( E[X | Y] = \frac{Y}{2} \) and \( E[Y] = \frac{\ell}{2} \). So then:
\[ E[X] = E[Y/2] = \frac{1}{2} E[Y] = \frac{1}{2} \cdot \frac{\ell}{2} = \frac{\ell}{4} \]

(b) We use the Law of Total Variance to find \( \text{var}(X) \):
\[ \text{var}(X) = E[\text{var}(X | Y)] + \text{var}(E[X | Y]). \]
Recall that the variance of a uniform random variable distributed over \([a, b]\) is \((b - a)^2 / 12\). Since \( Y \) is uniformly distributed over \([0, \ell]\), we have
\[
\text{var}(Y) = \frac{\ell^2}{12}, \\
\text{var}(X | Y) = \frac{Y^2}{12}.
\]
We know that \( E[X | Y] = Y/2 \), and so
\[
\text{var}(E[X | Y]) = \text{var}(Y/2) = \frac{1}{4} \text{var}(Y) = \frac{\ell^2}{48}.
\]
Also,
\[
E[\text{var}(X | Y)] = E\left[ \frac{Y^2}{12} \right] \\
= \int_0^\ell \frac{y^2}{12} f_Y(y) dy \\
= \frac{1}{12} \cdot \frac{1}{\ell} \int_0^\ell y^2 dy \\
= \frac{\ell^2}{36}.
\]
Combining these results, we obtain
\[
\text{var}(X) = E[\text{var}(X | Y)] + \text{var}(E[X | Y]) = \frac{\ell^2}{36} + \frac{\ell^2}{48} = \frac{7\ell^2}{144}.
\]

3. Let \( X_i \) denote the number of widgets in the \( i^{th} \) box. Then \( T = \sum_{i=1}^N X_i \).
\[
E[T] = E[E[\sum_{i=1}^N X_i | N]] \\
= E[\sum_{i=1}^N E[X_i | N]] \\
= E[\sum_{i=1}^N E[X]] \\
= E[X] \cdot E[N] = 100.
\]
and,

\[
\text{var}(T) = \mathbb{E}[\text{var}(T|N)] + \text{var}(\mathbb{E}[T|N]) \\
= \mathbb{E}\left[\text{var}\left(\sum_{i=1}^{N} X_i|N\right)\right] + \text{var}\left(\mathbb{E}\left[\sum_{i=1}^{N} X_i|N\right]\right) \\
= \mathbb{E}[N\text{var}(X)] + \text{var}(N\mathbb{E}[X]) \\
= (\text{var}(X))\mathbb{E}[N] + (\mathbb{E}[X])^2 \text{var}(N) \\
= 16 \cdot 10 + 100 \cdot 16 = 1760.
\]