Recitation 15 Solutions
October 28, 2010

1. (a) Let $X$ be the time until the first bulb failure. Let $A$ (respectively, $B$) be the event that the first bulb is of type $A$ (respectively, $B$). Since the two bulb types are equally likely, the total expectation theorem yields

$$E[X] = E[X|A]P(A) + E[X|B]P(B) = 1 \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \frac{2}{3}.$$

(b) Let $D$ be the event of no bulb failures before time $t$. Using the total probability theorem, and the exponential distributions for bulbs of the two types, we obtain

$$P(D) = P(D|A)P(A) + P(D|B)P(B) = \frac{1}{2} e^{-t} + \frac{1}{2} e^{-3t}.$$

(c) We have

$$P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{\frac{1}{2} e^{-t}}{\frac{1}{2} e^{-t} + \frac{1}{2} e^{-3t}} = \frac{1}{1 + e^{-2t}}.$$

(d) The lifetime of the first type-A bulb is $X_A$, with PDF given by:

$$f_{X_A}(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Let $Y$ be the total lifetime of two type-B bulbs. Because the lifetime of each type-B bulb is exponential with $\lambda = 3$, the sum $Y$ has an Erlang distribution of order 2 with $\lambda = 3$. Its PDF is:

$$f_Y(y) = \begin{cases} 9y e^{-3y} & y \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$P(G) = P(Y \geq X_A)$$

$$= \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{y} f_{X_A}(x) dx dy$$

$$= \int_{0}^{\infty} 9ye^{-3y} \int_{0}^{y} e^{-x} dx dy = 9 \int_{0}^{\infty} ye^{-3y} - e^{-y} \bigg|_{x=0}^{x=y} dy$$

$$= 9 \int_{0}^{\infty} ye^{-3y} - ye^{-4y} dy = 9 \left( -\frac{1}{3} ye^{-3y} - \frac{1}{9} e^{-3y} + \frac{1}{4} ye^{-4y} + \frac{1}{16} e^{-4y} \right) \bigg|_{y=\infty}^{y=0}$$

$$= 9 \left( \frac{1}{9} - \frac{1}{16} \right) = \frac{7}{16}$$

A simpler solution involving no integrals is as follows:

The bulb failure times of interest (1st type-A, 2nd type-B) may be thought of as the arrival
times of two independent Poisson processes of rate $\lambda_A = 1$ and $\lambda_B = 3$. We may imagine that these two processes were split from a joint Poisson process of rate $\lambda_A + \lambda_B$, where the splitting probabilities for each arrival are $P(A) = \frac{\lambda_A}{\lambda_A + \lambda_B} = 1/4$ to process $A$ and $P(B) = \frac{\lambda_B}{\lambda_A + \lambda_B} = 3/4$ to process $B$. Now we may just focus on whether arrivals to the joint process go to process $A$ or to process $B$. Each arrival to the joint process corresponds to an independent trial. There are two possible outcomes: the arrival is handed to process $A$ with probability $P(A)$ or the arrival is handed to process $B$ with probability $P(B)$. Then our event of interest occurs when either the first arrival goes to $A$, or the first arrival goes to $B$ followed by the second going to $A$. So the corresponding probability is

$$P(A \text{ or } BA) = P(A) + P(BA) = P(A) + P(B)P(A) = \frac{7}{16}$$

(e) Let $V$ be the total period of illumination provided by type-B bulbs while the process is in operation. Let $N$ be the number of light bulbs, out of the first 12, that are of type-B. Let $X_i$ be the period of illumination from the $i$th type-B bulb. We then have $V = Y_1 + \cdots + Y_N$. Note that $N$ is a binomial random variable, with parameters $n = 12$ and $p = 1/2$, so that

$$E[N] = 6, \quad \text{var}(N) = \frac{1}{2} \cdot \frac{1}{2} = 3.$$  

Furthermore, $E[X_i] = 1/3$ and $\text{var}(X_i) = 1/9$. Using the formulas for the mean and variance of the sum of a random number of random variables, we obtain

$$E[V] = E[N]E[X_i] = 2,$$

and

$$\text{var}(V) = \text{var}(X_i)E[N] + (E[X_i])^2\text{var}(N) = \frac{1}{9} \cdot 6 + \frac{1}{9} \cdot 3 = 1.$$  

(f) Using the notation in parts (a)-(c), and the result of part (c), we have

$$E[T|D] = t + E[T-t|D \cap A]P(A|D) + E[T-t|D \cap B]P(B|D)$$

$$= t + 1 \cdot \frac{1}{1 + e^{-2t}} + \frac{1}{3} \left(1 - \frac{1}{1 + e^{-2t}}\right)$$

$$= t + \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{1 + e^{-2t}}.$$  

2. (a) The total arrival process corresponds to the merging of two independent Poisson processes, and is therefore Poisson with rate $\lambda = \lambda_A + \lambda_B = 7$. Thus, the number $N$ of jobs that arrive in a given three-minute interval is a Poisson random variable, with $E[N] = 3\lambda = 21$, $\text{var}(N) = 21$, and PMF

$$p_N(n) = \frac{(21)^n e^{-21}}{n!}, \quad n = 0, 1, 2, \ldots.$$  

(b) Each of these 10 jobs has probability $\lambda_A/(\lambda_A + \lambda_B) = 3/7$ of being type $A$, independently of the others. Thus, the binomial PMF applies and the desired probability is equal to

$$\binom{10}{3} \left(\frac{3}{7}\right)^3 \left(\frac{4}{7}\right)^7.$$
(c) Each future arrival is of type A with probability \( \frac{\lambda_A}{\lambda_A + \lambda_B} = \frac{3}{7} \) of being type A, independently of the others. Thus, the number \( K \) of arrivals until the first type A arrival is geometric with parameter \( \frac{3}{7} \). The number of type B arrivals before the first type A arrival is equal to \( K - 1 \), and its PMF is similar to a geometric, except that it is shifted by one unit to the left. In particular,

\[
p_K(k) = \left( \frac{3}{7} \right) \left( \frac{4}{7} \right)^k, \quad k = 0, 1, 2, \ldots .
\]

3. The event \( \{ X < Y < Z \} \) can be expressed as \( \{ X < \min\{Y, Z\} \} \cap \{ Y < Z \} \). Let \( Y \) and \( Z \) be the 1st arrival times of two independent Poisson processes with rates \( \mu \) and \( \nu \). By merging the two processes, it should be clear that \( Y < Z \) if and only if the first arrival of the merged process comes from the original process with rate \( \mu \), and thus

\[
P(Y < Z) = \frac{\mu}{\mu + \nu}.
\]

Let \( X \) be the 1st arrival time of a third independent Poisson process with rate \( \lambda \). Now \( \{ X < \min\{Y, Z\} \} \) if and only if the first arrival of the Poisson process obtained by merging the two processes with rates \( \lambda \) and \( \mu + \nu \) comes from the original process with rate \( \lambda \), and thus

\[
P(X < \min\{Y, Z\}) = \frac{\lambda}{\lambda + \mu + \nu}.
\]

Note that the event \( \{ X < \min\{Y, Z\} \} \) is independent of the event \( \{ Y < Z \} \), as the time of the first arrival of the merged process with rate \( \mu + \nu \) is independent of whether that first arrival comes from the process with rate \( \mu \) or the process with rate \( \nu \). Hence,

\[
P(X < Y < Z) = P(X < \min\{Y, Z\}) \cdot P(Y < Z)
= \frac{\lambda \mu}{(\lambda + \mu + \nu)(\mu + \nu)}.
\]