Recitation 21 Solutions  
November 23, 2010

1. (a) To use the Markov inequality, let $X = \sum_{i=1}^{10} X_i$. Then,

$$E[X] = 10E[X_i] = 5,$$

and the Markov inequality yields

$$P(X \geq 7) \leq \frac{5}{7} = 0.7142.$$

(b) Using the Chebyshev inequality, we find that

$$2P(X - 5 \geq 2) = P(|X - 5| \geq 2) \leq \frac{\text{var}(X)}{4} = \frac{10/12}{4}$$

$$P(X - 5 \geq 2) \leq \frac{5}{48} = 0.1042.$$

(c) Finally, using the Central Limit Theorem, we find that

$$P\left(\sum_{i=1}^{10} X_i \geq 7\right) = 1 - P\left(\sum_{i=1}^{10} X_i \leq 7\right)$$

$$= 1 - P\left(\frac{\sum_{i=1}^{10} X_i - 5}{\sqrt{10/12}} \leq \frac{7 - 5}{\sqrt{10/12}}\right)$$

$$\approx 1 - \Phi(2.19)$$

$$\approx 0.0143.$$

2. Check online solutions.

3. (a) If we interpret $X_i$ as the number of arrivals in an interval of length 1 in a Poisson process of rate 1, then, $S_n = X_1 + \cdots + X_n$ can be seen as the number of arrivals in an interval of length $n$ in the Poisson process of rate 1. Therefore, $S_n$ is a Poisson random variable with mean and variance equal to $n$.

(b) We use the random variables $X_1, \ldots, X_n$ and the random variable $S_n = X_1 + \cdots + X_n$. Denoting by $Z$ the standard normal, and applying the central limit theorem, we have for
large $n$

\[
P(S_n = n) = P(n - 1/2 < S_n < n + 1/2)
= P\left(\frac{-1}{2\sqrt{n}} < \frac{S_n - n}{\sqrt{n}} \leq \frac{1}{2\sqrt{n}}\right)
\approx P\left(\frac{-1}{2\sqrt{n}} < Z \leq \frac{1}{2\sqrt{n}}\right)
= \frac{1}{\sqrt{2\pi}} \int_{-1/2\sqrt{n}}^{1/2\sqrt{n}} e^{-z^2/2} dz
\approx \frac{1}{\sqrt{2\pi}} \left| e^{-z^2/2} \right|_{z=0}
= \frac{1}{\sqrt{2\pi n}}
\]

where the first equation follows from the fact that $S_n$ takes integer values, the first approximation is suggested by the central limit theorem, and the second approximation uses the fundamental theorem of calculus (the value of a definite integral over a small interval is equal to the length of the interval times the integrand evaluated at some point within the interval). Since $S_n$ is Poisson with mean $n$, we have

\[
P(S_n = n) = e^{-n} \frac{n^n}{n!},
\]

and by combining the preceding relations, we see that $n! \approx n^n e^{-n} \sqrt{2\pi n} = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$.

One may show that

\[
limit_{n \to \infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1,
\]

so the relative error of the approximation tends to 0 as $n \to \infty$. A more precise estimate is that

\[n! = n^n e^{-n} \sqrt{2\pi n} \cdot e^{\lambda_n},\]

where

\[
\frac{1}{12n + 1} < \lambda_n < \frac{1}{12n}.
\]

However, one cannot derive these relations from the central limit theorem.

Note that the form of the approximation was first discovered by de Moivre in the form $n! \approx n^{n+1/2} e^{-n} \cdot (\text{constant})$, and gave a complicated expression for the constant. De Moivre’s friend Stirling subsequently showed that the constant has the simple form $\sqrt{2\pi}$.