1. (a) Normalization of the distribution requires:

\[ 1 = \sum_{k=0}^{\infty} p_K(k; \theta) = \sum_{k=0}^{\infty} e^{-k/\theta} \frac{1}{Z(\theta)} \sum_{k=0}^{\infty} e^{-k/\theta} = \frac{1}{Z(\theta)} \left( 1 - e^{-1/\theta} \right), \]

so \( Z(\theta) = \frac{1}{1-e^{-1/\theta}} \).

(b) Rewriting \( p_K(k; \theta) \) as:

\[ p_K(k; \theta) = \frac{\left( e^{-1/\theta} \right)^k}{Z(\theta)} \left( 1 - e^{-1/\theta} \right), \quad k = 0, 1, \ldots \]

the probability distribution for the photon number is a geometric probability distribution with probability of success \( p = 1 - e^{-1/\theta} \), and it is shifted with 1 to the left since it starts with \( k = 0 \). Therefore the photon number expectation value is

\[ \mu_K = \frac{1}{p} - 1 = \frac{1}{1 - e^{-1/\theta}} - 1 = \frac{1}{e^{1/\theta} - 1} \]

and its variance is

\[ \sigma_K^2 = \frac{1-p}{p^2} = \frac{e^{-1/\theta}}{(1 - e^{-1/\theta})^2} = \mu_K + \mu_K. \]

(c) The joint probability distribution for the \( k_i \) is

\[ p_K(k_1, \ldots, k_n; \theta) = \frac{1}{Z(\theta)^n} \Pi_{i=1}^{n} e^{-k_i/\theta} = \frac{1}{Z(\theta)^n} e^{-\frac{1}{\theta} \sum_{i=1}^{n} k_i}. \]

The log likelihood is \(-n \cdot \log Z(\theta) - 1/\theta \sum_{i=1}^{n} k_i \).

We find the maxima of the log likelihood by setting the derivative with respect to the parameter \( \theta \) to zero:

\[ \frac{d}{d\theta} \log p_K(k_1, \ldots, k_n; \theta) = -n \cdot \frac{e^{-1/\theta}}{\theta^2 (1 - e^{-1/\theta})} + \frac{1}{\theta^2} \sum_{i=1}^{n} k_i = 0 \]

or

\[ \frac{1}{e^{1/\theta} - 1} = \frac{1}{n} \sum_{i=1}^{n} k_i = s_n. \]

For a hot body, \( \theta \gg 1 \) and \( \frac{1}{e^{1/\theta} - 1} \approx \theta \), we obtain

\[ \theta \approx \frac{1}{n} \sum_{i=1}^{n} k_i = s_n. \]

Thus the maximum likelihood estimator \( \hat{\Theta}_n \) for the temperature is given in this limit by the sample mean of the photon number.
\( \hat{\theta}_n = \frac{1}{n} \sum_{i=1}^{n} K_i. \)

(d) According to the central limit theorem, the sample mean for large enough \( n \) (in the limit) approaches a Gaussian distribution with standard deviation our root mean square error

\[ \sigma_{\hat{\theta}_n} = \frac{\sigma_K}{\sqrt{n}}. \]

To allow only for 1% relative root mean square error in the temperature, we need \( \frac{\sigma_K}{\sqrt{n}} < 0.01\mu_K \). With \( \sigma_K^2 = \mu_K + \mu_K \) it follows that

\[ \sqrt{n} > \frac{\sigma_K}{0.01\mu_K} = 100\sqrt{\mu_K^2 + \mu_K} = 100\sqrt{1 + \mu_K}. \]

In general, for large temperatures, i.e. large mean photon numbers \( \mu_K \gg 1 \), we need about 10,000 samples.

(e) The 95% confidence interval for the temperature estimate for the situation in part (d), i.e.

\[ \sigma_{\hat{\theta}_n} = \frac{\sigma_K}{\sqrt{n}} = 0.01\mu_K, \]

is

\[ [\hat{K} - 1.96\sigma_K, \hat{K} + 1.96\sigma_K] = [\hat{K} - 0.0196\mu_K, \hat{K} + 0.0196\mu_K]. \]

2. (a) Using the regression formulas of Section 9.2, we have

\[ \hat{\theta}_1 = \frac{\sum_{i=1}^{5} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{5} (x_i - \bar{x})^2}, \quad \hat{\theta}_0 = \bar{y} - \hat{\theta}_1 \bar{x}, \]

where

\[ \bar{x} = \frac{1}{5} \sum_{i=1}^{5} x_i = 4.94, \quad \bar{y} = \frac{1}{5} \sum_{i=1}^{5} y_i = 134.38. \]

The resulting ML estimates are

\( \hat{\theta}_1 = 40.53, \quad \hat{\theta}_0 = -65.86. \)

(b) Using the same procedure as in part (a), we obtain

\[ \hat{\theta}_1 = \frac{\sum_{i=1}^{5} (x_i^2 - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{5} (x_i^2 - \bar{x})^2}, \quad \hat{\theta}_0 = \bar{y} - \hat{\theta}_1 \bar{x}, \]
where
\[ \bar{x} = \frac{1}{5} \sum_{i=1}^{5} x_i^2 = 33.60, \quad \bar{y} = \frac{1}{5} \sum_{i=1}^{5} y_i = 134.38. \]
which for the given data yields
\[ \hat{\theta}_1 = 4.09, \quad \hat{\theta}_0 = -3.07. \]
Figure 1 shows the data points \((x_i, y_i), i = 1, \ldots, 5\), the estimated linear model
\[ y = 40.53x - 65.86, \]
and the estimated quadratic model
\[ y = 4.09x^2 - 3.07. \]