Tutorial 3: Solutions

1. In general we have that $E[aX + bY + c] = aE[X] + bE[Y] + c$. Therefore,

$$E[Z] = 2 \cdot E[X] - 3 \cdot E[Y].$$

For the case of independent random variables, we have that if $Z = a \cdot X + b \cdot Y$, then

$$\text{var}(Z) = a^2 \cdot \text{var}(X) + b^2 \cdot \text{var}(Y).$$

Therefore, $\text{var}(Z) = 4 \cdot \text{var}(X) + 9 \cdot \text{var}(Y)$.

2. See online solutions.

3. (a) We can find $c$ knowing that the probability of the entire sample space must equal 1.

$$1 = \sum_{x=1}^{3} \sum_{y=1}^{3} p_{X,Y}(x, y)$$

$$= c + c + 2c + 2c + 4c + 3c + c + 6c$$

$$= 20c$$

Therefore, $c = \frac{1}{20}$.

(b) $p_Y(2) = \sum_{x=1}^{3} p_{X,Y}(x, 2) = 2c + 0 + 4c = 6c = \frac{3}{10}$.

(c) $Z = XY^2$

$$E[Z \mid Y = 2] = E[XY^2 \mid Y = 2]$$

$$= E[2X^2 \mid Y = 2]$$

$$= 2E[X^2 \mid Y = 2]$$

$$p_{X \mid Y}(x \mid 2) = \frac{p_{X,Y}(x, 2)}{p_Y(2)}.$$

Therefore, $p_{X \mid Y}(x \mid 2) = \left\{ \begin{array}{ll}
\frac{1}{10} & \text{if } x = 1 \\
\frac{3}{10} & \text{if } x = 3 \\
\frac{1}{5} & \text{if } x = 2 \\
0 & \text{otherwise} \end{array} \right.$

$$E[Z \mid Y = 2] = 2 \sum_{x=1}^{3} x^2 p_{X \mid Y}(x \mid 2)$$

$$= 2 \left( (1^2) \cdot \frac{1}{3} + (3^2) \cdot \frac{2}{3} \right)$$

$$= \frac{38}{3}$$
(d) Yes. Given $X \neq 2$, the distribution of $X$ is the same given $Y = y$.

$$P(X = x \mid Y = y, X \neq 2) = P(X = x \mid X \neq 2).$$

For example,

$$P(X = 1 \mid Y = 1, X \neq 2) = P(X = 1 \mid Y = 3, X \neq 2) = P(X = 1 \mid X \neq 2) = \frac{1}{3}.$$

(e) $p_{Y|X}(y \mid 2) = \frac{p_{X,Y}(2,y)}{p_X(2)}$.

$p_X(2) = \sum_{y=1}^{3} p_{X,Y}(2,y) = c + 0 + c = 2c = \frac{10}{10}$.

Therefore,

$$p_{Y|X}(y \mid 2) = \begin{cases} 
\frac{1/20}{1/10} = \frac{1}{2} & \text{if } y = 1 \\
\frac{1/20}{1/10} = \frac{1}{2} & \text{if } y = 3 \\
0 & \text{otherwise}
\end{cases}$$

$$E[Y^2 \mid X = 2] = \sum_{y=1}^{3} y^2 p_{Y|X}(y \mid 2) = (1^2) \cdot \frac{1}{2} + (3^2) \cdot \frac{1}{2} = 5.$$  

$$E[Y \mid X = 2] = \sum_{y=1}^{3} y p_{Y|X}(y \mid 2) = 1 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = 2.$$  

$$\text{var}(Y \mid X = 2) = E[Y^2 \mid X = 2] - E[Y \mid X = 2]^2 = 5 - 2^2 = 1.$$