6.041/6.431 LECTURE I

September 3, 2008
John Wyatt

- 6.041/6.431 – Probability
  Lecturer: John Wyatt
  Head TA: Paul Njoroge

- Pick up **and read** course information handout. Pull out the last page, fill it out and **turn it in at the end of lecture** to register.

- Lectures and recitations are mandatory. Tutorials are optional. Most students find the tutorials essential for learning the material.

- Pick up 1 copy of today's lecture.

- Pick up one copy of Problem set 1. Homeworks will be available on the web each Wednesday. Solution sets will appear on the web the following Wednesday. Problem sets must be turned in at the beginning of lecture outside 34-101 one week after they are handed out, between 12:00 and 12:15. Homeworks handed in after 12:15 will not be graded.

- **Good** Text – Bertsekas and Tsitsiklis, "Introduction to Probability," 2^{nd} edition. It is in stock at the Coop. The 1^{st} edition will not be useful for many parts of the course.

PROBABILITY PROBLEM I: THE BABY GIRL PROBLEM

In a certain country every child is a boy or girl with exactly 50% probability, regardless of the sex of any other children in that (or any other) family. Every family wants to raise a baby girl, so the country has adopted a policy that every family continues to give birth to more children until they have their first baby girl, and then that family must stop. In this fertile population, no family is unable to have another child if they try.

HYPOTHESIS: After many generations of this policy, the expected number of girls in the population becomes larger than the expected number of boys.

1. **ARGUMENT IN FAVOR:** In the usual individualistic family planning practices, half the population is female on average, although some families have only boys. But in the country above, every family with children has a girl.

Therefore the average number of girls born must be greater than the average number of boys.

2. **ARGUMENT AGAINST:** Choosing or not choosing to have another child does not alter the probability that the next child is female.

Therefore, the policy in that country cannot make the average number of girls exceed the average number of boys.

PROBABILITY PROBLEM II:
THE DOUBLE OR QUARTER GAME

In the double-or-quarter game, you initially put in \$1.00. After every toss of a fair coin, your money doubles if the coin comes up heads, but you lose three quarters of your fortune if the coin comes up tails. You continue to bet your remaining fortune on the next toss.

HYPOTHESIS: This is a good game to play in the long run because your fortune rises, on average, with each toss.

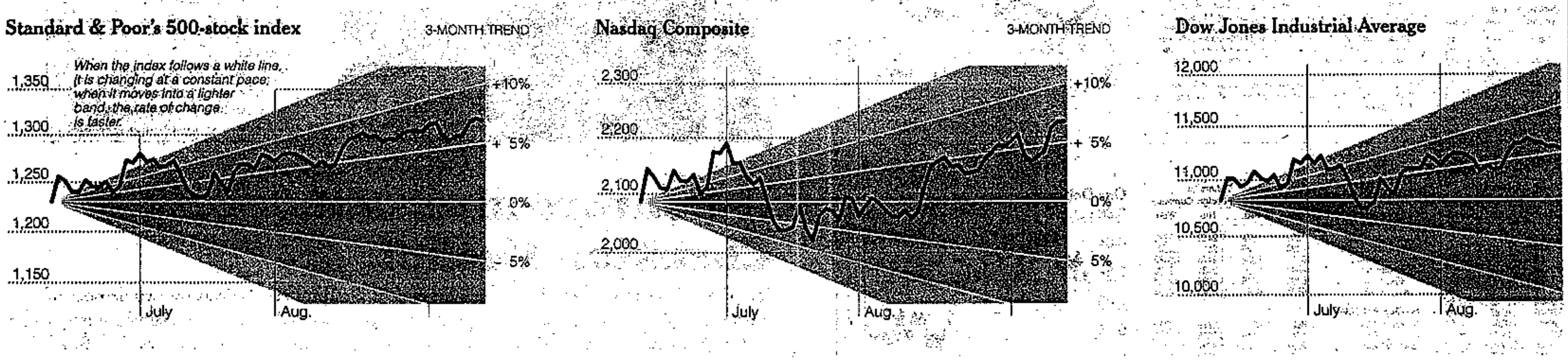
ARGUMENT IN FAVOR: After one toss you have \$2.00 if heads comes up, otherwise you have \$ 0.25. Each is equally likely, so your average fortune is the average of these two numbers, \$1.125, i.e., 9/8 dollars. A similar multiplication occurs with each subsequent toss, so your expected wealth after n tosses is $\$(9/8)^n$. On average you'll get rich if you continue with this game.

ARGUMENT AGAINST: Who are you kidding? It takes exactly two wins to cancel one loss. For example, out of 99 tosses, you have to have at least 66 heads to break even! In the long run the probability of having at least twice as many wins as losses will have to become extremely small, so you'll almost certainly lose. This is a game for fools !

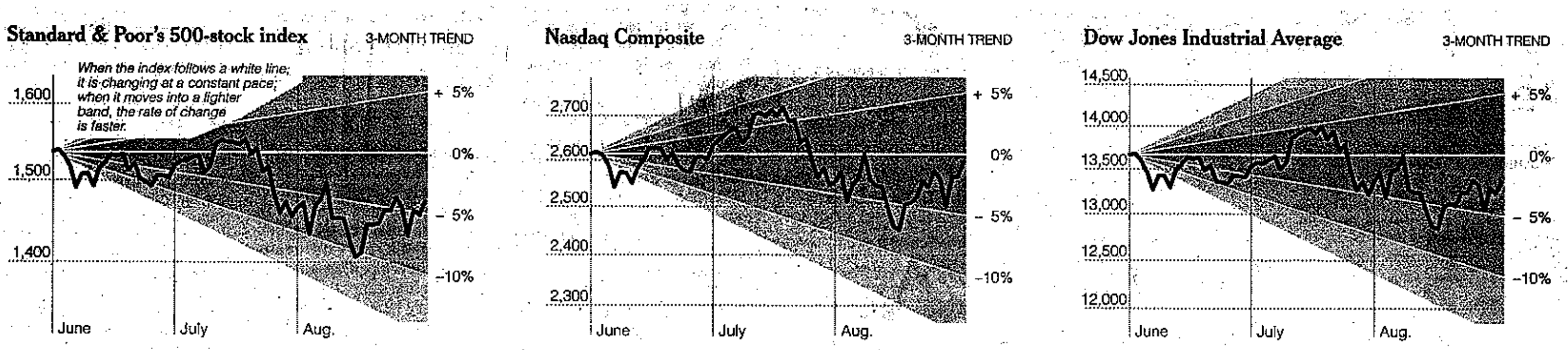
PROBABILITY (MODELLING) PROBLEM III: STOCK MARKET INDICES

The New York Times Financial Section often includes plots like the ones below, showing the change in a market index over the three month period up though the previous day. The sloping lines indicate possible constant rates of growth or loss for the index in question. The plot usually wanders outside of the sloping lines altogether for short times but falls within them for longer times. Why is this behavior typical?

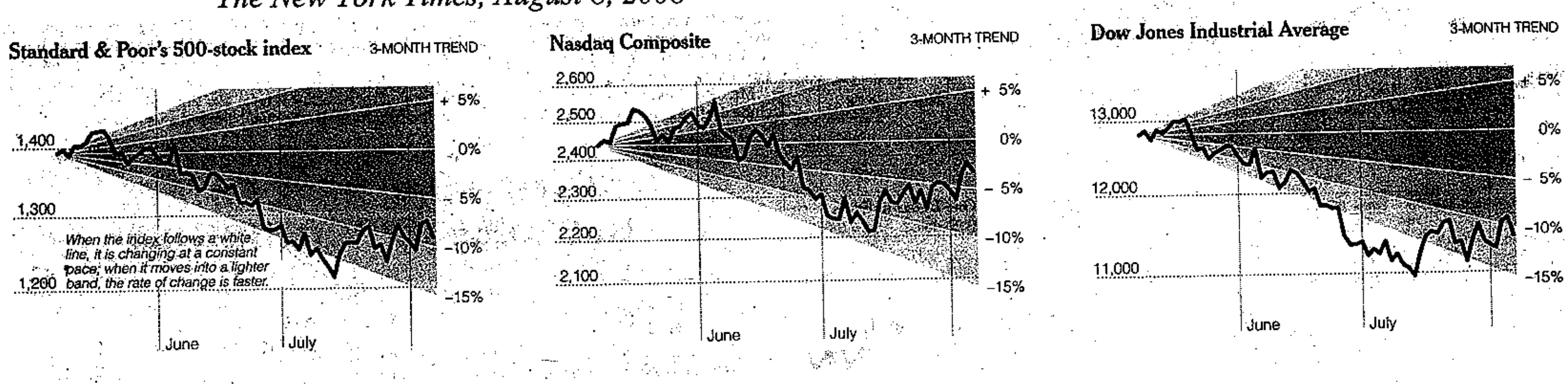
The New York Times September 15, 2006



The New York Times, September 1, 2007



The New York Times, August 8, 2008



Sample space

- List of possible outcomes
- List of must be:

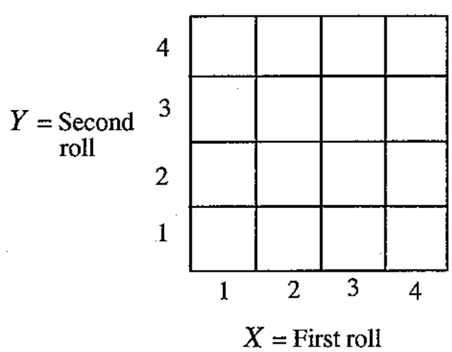
– Mutually exclusive

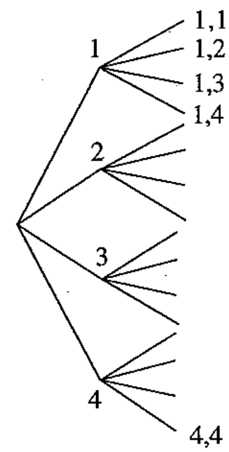
– Collectively exhaustive

– At the “right” granularity

Sample space examples

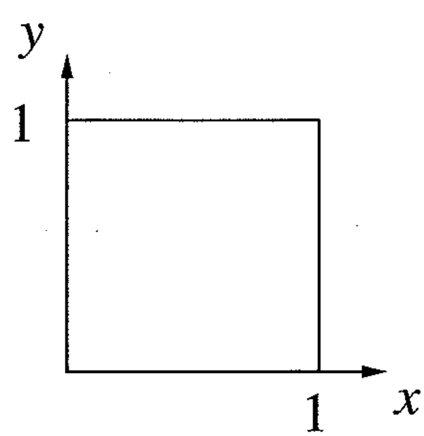
- Two rolls of a tetrahedral die
  - Sample space vs. sequential description





- A continuous sample space:

(x, y) such that $0 \leq x, y \leq 1$



Axioms of probability

- **Sample Space:** a list of all possible outcomes
- **Event:** a subset of the sample space
- Every outcome is an event, and certain sets of outcomes are events
- Probability is assigned to events

Axioms:

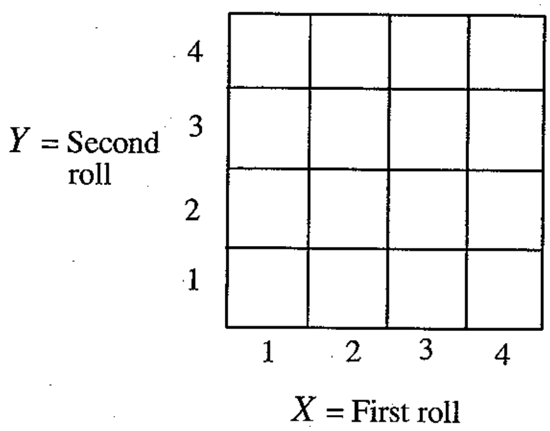
1. $P(A) \geq 0$
2. P(universe) = 1
3. If $A \cap B = \varnothing$,

   Then $P(A \cup B) = P(A) + P(B)$

- $P(\{s_1, s_2, \ldots, s_k\}) = P(s_1) + \cdots + P(s_k)$

- Axiom 3 needs strengthening
- Do weird sets have probabilities?

Example



- Let every possible outcome have probability 1/16
- $P(X=1)=$
- Let $Z=\min(X,Y)$
- $P(Z=1)=$
- $P(Z=2)=$
- $P(Z=3)=$
- $P(Z=4)=$

Discrete uniform law

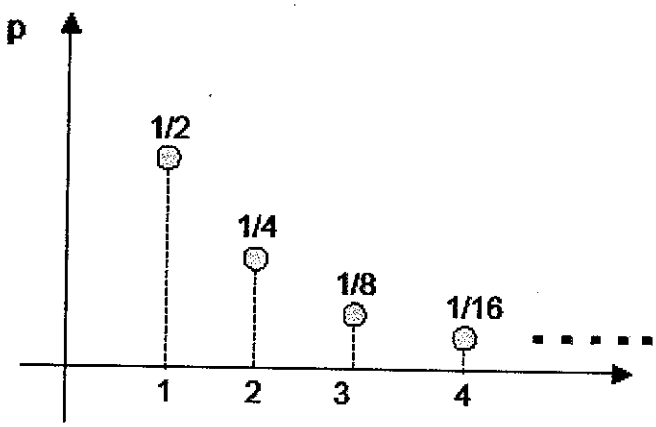
- Let all sample points be equally likely
- Then,

$$P(A) = \frac{\text{number of elements of } A}{\text{total number of sample points}}$$

- Just count…

A word about infinite sample spaces

- Sample space: $\{1,2,...\}$
  - We are given $P(n)=2^{-n}, n=1,2,\ldots$
  - Find **P** (outcome is even)



- Solution:

$$P(\{2,4,6,\ldots\})=P(2)+P(4)+\cdots$$
$$=\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}+\cdots=\frac{1}{3}$$

- Axiom needed:

If $A_1, A_2\ldots$ are (a possibly infinite collection of) disjoint events, then:

$$P(A_1\cup A_2\cup\cdots)=P(A_1)+P(A_2)+\cdots$$