LECTURE 9

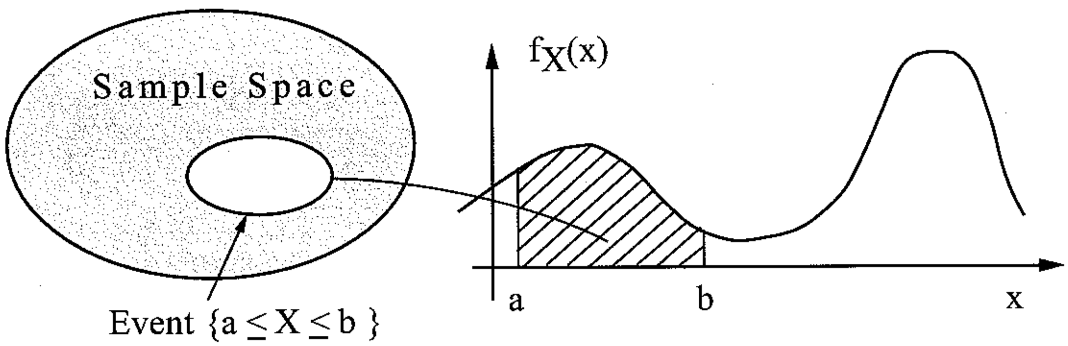
October 6, 2008

- **Readings:** Sections 3.4-3.5

Outline

- PDF review
- Multiple random variables
  - conditioning
  - independence
- Examples

Continuous r.v.'s and pdf's



$$\mathbf{P}(a \le X \le b) = \int_a^b f_X(x)\,dx$$

- $\mathbf{P}(x \le X \le x + \delta) \approx f_X(x) \cdot \delta$
- $\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$

Summary of concepts

$p_X(x)$ $\qquad$ $f_X(x)$

$$F_X(x)$$

$$\mathbf{E}[X], \ \mathrm{var}(X)$$

$p_{X,Y}(x,y)$ $\qquad$ $f_{X,Y}(x,y)$

$p_{X|Y}(x \mid y)$ $\qquad$ $f_{X|Y}(x \mid y)$

Joint PDF $f_{X,Y}(x,y)$

$$\mathbf{P}(A) = \int\int_A f_{X,Y}(x,y)\,dx\,dy$$

- Interpretation:

$$\mathbf{P}(x \le X \le x+\delta,\ y \le Y \le y+\delta) \approx f_{X,Y}(x,y)\cdot\delta^2$$

- Expectations:

$$\mathbf{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$$

- From the joint to the marginal:

$$f_X(x)\cdot\delta \approx \mathbf{P}(x \le X \le x+\delta) =$$

- X and Y are called independent if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

Conditioning

- Recall

$$\mathbf{P}(x \le X \le x + \delta) \approx f_X(x) \cdot \delta$$

- By analogy, would like:

$$\mathbf{P}(x \le X \le x + \delta \mid Y \approx y) \approx f_{X|Y}(x \mid y) \cdot \delta$$

- This leads us to the definition:

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

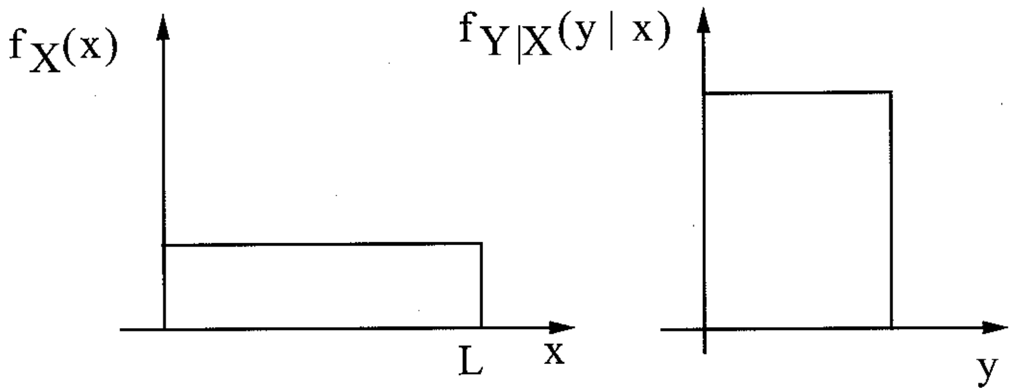
- Conditional is a "section" of the joint pdf (normalized)

- If independent, $f_{X,Y} = f_X f_Y$, we obtain

$$f_{X|Y}(x|y) = f_X(x)$$

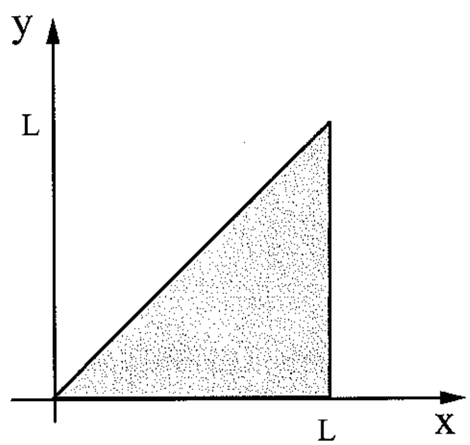
Stick-breaking example

- Break a stick of length L twice, at uniformly chosen random points
  - X, Y: point of first and second break



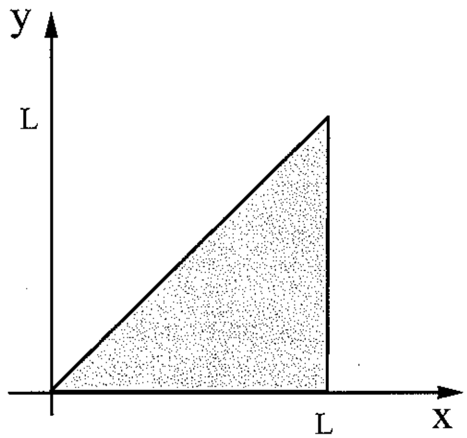
$$f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y \mid x) =$$

on the set:



$$\mathbf{E}[Y \mid X = x] = \int y f_{Y|X}(y \mid X = x)\, dy =$$

$$f_{X,Y}(x, y) = \frac{1}{Lx}, \qquad 0 \le y \le x \le L$$



$$\begin{aligned} f_Y(y) &= \int f_{X,Y}(x, y)\, dx \\ &= \int_y^L \frac{1}{Lx}\, dx \\ &= \frac{1}{L} \log \frac{L}{y}, \qquad 0 \le y \le L \end{aligned}$$

$$\mathbf{E}[Y] = \int_0^L y f_Y(y)\, dy = \int_0^L y \frac{1}{L} \log \frac{L}{y}\, dy = \frac{L}{4}$$