Problem Set 4
Due: March 5, 2008

1. It's yet another warm Sunday afternoon (in our 6.041 fantasies) and you're taking a break from probability by feeding the birds in the park. As soon as the birds spot you, three of them come over: a pigeon, a robin and a turkey. If you allot your k seeds amongst the three birds, with each distinct allotment having equal probability, what is the probability that the turkey goes hungry (i.e., gets zero seeds)? Compute this probability for both (i) the case that the seeds are distinguishable; and (ii) the case that the seeds are indistinguishable.

   Hint: When the seeds are indistinguishable consider the following scenario: place the seeds in a line and insert two separators. The seeds to the left of the first separator will be fed to the turkey, to the right of the second separator to the robin, and in between separators to the pigeon. Each way of positioning the two separators corresponds to a different way of assigning the seeds to three birds. How many ways are there to arrange these $k+2$ objects (k seeds and two separators) given that the separators are indistinguishable and the seeds are indistinguishable? Furthermore, the event that the turkey goes hungry corresponds to at least one of the separators coming before all the seeds. Therefore one separator must come first, but the second separator can take on any of the $k+1$ possible locations. So, how many ways are there for the turkey to go hungry?

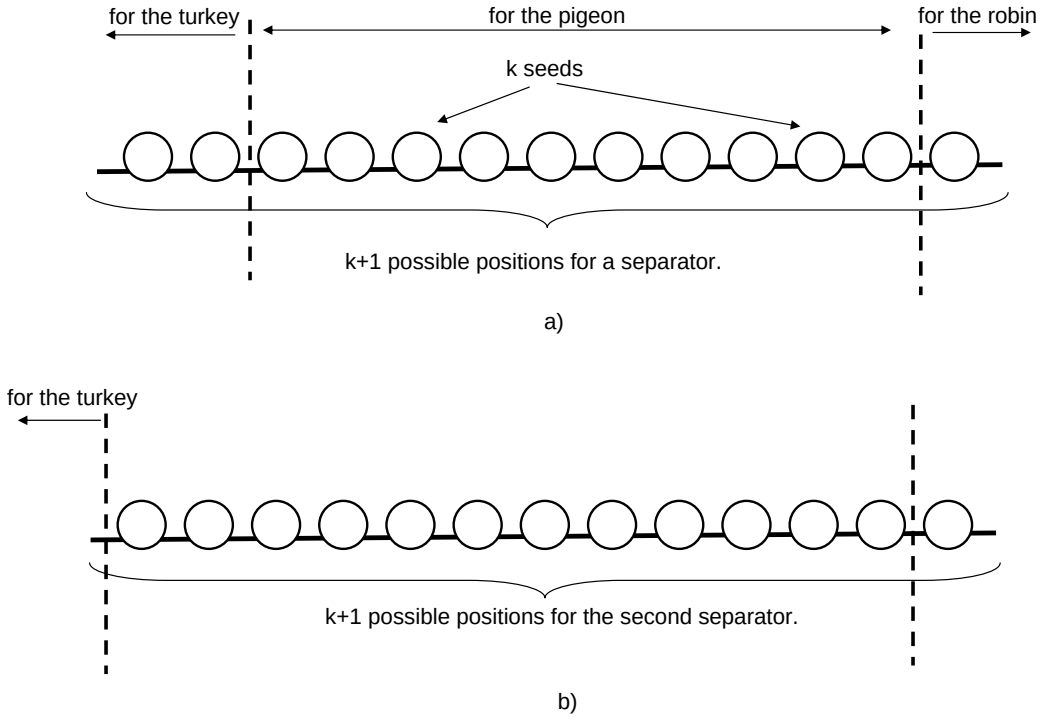

Figure 1: Problem 1 – The Case of Indistinguishable Seeds

2. Your probability class has 250 undergraduate students and 50 graduate students. Grades are assigned randomly, with undergraduate students getting an A with probability 1/3 and graduate students getting an A with probability 1/2. Let X be the number of students that get an A. Calculate $\mathbf{E}[X]$ and $\text{var}(X)$ by viewing X as a sum of random variables, whose statistics are easily calculated. (*Hint:* Your answer for $\mathbf{E}[X]$ should match the value obtained in the previous problem set.)

3. You are lost in a labyrinth full of mythical creatures. In order to purchase a magical compass that will guide you out, you need five pieces of axinite. Since the inhabitants of the labyrinth are enchanted by music, you decide to sing a few songs in hope of remuneration. For each song you sing, the mayor of the labyrinth awards you a piece of axinite if she happens to like the song. Suppose that the mayor likes each song with probability p, independently of all other songs.

   (a) Let K be the number of pieces of axinite you've acquired after singing 40 songs. Determine the PMF, expectation and variance of K.

   (b) Let M be the number of songs you need to sing *before* the first one for which you are given a piece of axinite. Determine the PMF, expectation and variance of M.

   (c) Let N be the number of songs you need to sing to acquire the five needed pieces of axinite. Determine the PMF of N. (You do not need to compute its expectation and variance.)

4. Still in the labyrinth, you've purchased the magic compass and have k pieces of axinite left. You decide to spend it on some souvenirs, but you need to choose how much of it to spend on books and how much on gadgets. You draw the pieces one by one from your purse and, for each piece, you choose with probability p to place it into your left pocket (in which case you will spend it on a book) or with probability $1-p$ into your right pocket (in which case you will spend it on a gadget), independently of your previous choices. Let R be the total number of pieces in your right pocket and L be the total number of pieces in your left pocket.

   (a) Determine the probability mass function for random variable R. Evaluate its expected value and variance.

   (b) Evaluate the probability that the first piece of axinite you drew out of your purse ends up being the only one in that pocket.

   (c) Evaluate the probability that at least one of your pockets contains exactly one piece of axinite.

   (d) Evaluate the expectation and the variance for the difference, $D = R - L$.

   (e) Given the first two pieces of axinite both went into your right pocket, find the conditional PMF, conditional expectation, and conditional variance for random variable R.

5. At Tony Laplace's House of Pizza, the customer doesn't choose toppings—lady luck does. Four toppings are available: (1) mushroom, (2) sausage, (3) pepperoni and (4) onion. Each pizza can have any combination of toppings, and topping i is included with probability q_i independent of whether that pizza includes other toppings. (That is, mushroom is included with probability q_1, sausage with probability q_2, etc.) On a day in which the number of pizzas sold is n, let N_i equal the number of pizzas sold with topping i. What is the joint PMF $p_{N_1,N_2,N_3,N_4}(n_1,n_2,n_3,n_4)$?

G1[†]. In your closet, you have m pairs of shoes (therefore, $2m$ shoes in total). After ten years each shoe got lost, independently, with probability p. Let N be the number of shoes remaining in your closet and let C be the number of complete pairs. (Assume no two shoes are alike, unless they belong to the same pair.) Find $\mathbf{E}[C \mid N = n]$.

Hint: Define X_i to be the random variable taking the value 1 or 0 depending on whether, after ten years, the left shoe of the ith pair is lost or not. Let Y_i be the corresponding random variable for the right shoe of the same pair. Finally, let $C = \sum_{i=1}^{m} X_i Y_i$.

[†]Required for 6.431; optional for 6.041