Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2008)

Practice Problems on Estimation Topics and Markov Chains: Solutions
May 12, 2008

Solutions for ECP are found on the course website.

Recommendations for ECP[1] on Estimation Topics: Found in new Chapter 6 Handout..

1. ECP Section 6.3 Problem 8, 9, and 12
2. ECP Section 6.4 Problem 17, 20, 22, and 23
3. ECP Section 6.6 Problem 36

Recommendations for ECP on Markov Chains: Found in Chapter 6 of Course Text.

1. ECP Section 6.3 Problem 9, 11, and 12
2. ECP Section 6.4 Problem 27, 28

Supplemental PSet problems for Markov Chains

1. (a) The recurrent classes are $\{1\}$ and $\{5, 6\}$. They are both aperiodic since the self transition probabilities are greater than 0, i.e., $p_{ii} > 0$ for $i = 1, 5, 6$.

   (b) Let a_i denote the probability of absortion into State 1 starting from state i. Then, it's clear that $a_1 = 1$ and $a_3 = 0$. Direct application of the Total Probability Theorem yields

   $$a_2 = \frac{1}{3}a_1 + \frac{1}{6}a_2 + \frac{1}{2}a_3 = \frac{1}{3} + \frac{1}{6}a_2.$$

   Therefore $a_2 = \frac{2}{5}$.
   Let b_i denote the absorption probability into $\{5, 6\}$. Note that $a_i + b_i = 1$. Therefore $b_2 = \frac{3}{5}$.

   (c) Note that we need to only compute the a_i's. We already have the values for $i = 1, 2, 3$ from Part (b). It's easy that verify that $a_4 = a_5 = a_6 = 0$, and hence $b_4 = b_5 = b_6 = 1$. The results are summarized in the following table.

i	1	2	3	4	5	6
a_i	1	2/5	0	0	0	0
b_i	0	3/5	1	1	1	1

   (d) Let π_j^i denote the steady state probability for state j starting from the recurrent class $\{i\}$. The steady state probabilities for the recurrent class $\{5, 6\}$ are obtained by solving the following balance equations:

[1] End of Chapter Problems

$$\begin{aligned}
\pi_5^{\{5,6\}} &= \frac{1}{2}\pi_6^{\{5,6\}} + \frac{1}{4}\pi_5^{\{5,6\}} \\
\pi_6^{\{5,6\}} &= \frac{3}{4}\pi_5^{\{5,6\}} + \frac{1}{2}\pi_6^{\{5,6\}} \\
1 &= \pi_5^{\{5,6\}} + \pi_6^{\{5,6\}}
\end{aligned}$$

Solving for the steady-state probabilities we get:

$$\pi_5^{\{5,6\}} = \frac{2}{5}$$

$$\pi_6^{\{5,6\}} = \frac{3}{5}$$

If the Markov chain starts in state 1, then $\pi_1^{\{1\}} = 1$.
Therefore,

$$\begin{aligned}
\lim_{n\to\infty} P(X_n = j|X_0 = i) &= \lim_{n\to\infty}\{P(X_n = j|X_0 = i,\text{absorption class is } \{1\}) \\
&\quad P(\text{absorption class is } \{1\}|X_0 = i) \\
&+ P(X_n = j|X_0 = i,\text{absorption class is } \{5,6\}) \\
&\quad P(\text{absorption class is } \{5,6\}|X_0 = i)\}
\end{aligned}$$

and

$$r_{ij}(\infty) = a_i\pi_j^{\{1\}} + b_i\pi_j^{\{5,6\}}.$$

$$r_{ij}(\infty) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2/5 & 0 & 0 & 0 & 6/25 & 9/25 \\ 0 & 0 & 0 & 0 & 2/5 & 3/5 \\ 0 & 0 & 0 & 0 & 2/5 & 3/5 \\ 0 & 0 & 0 & 0 & 2/5 & 3/5 \\ 0 & 0 & 0 & 0 & 2/5 & 3/5 \end{bmatrix}$$

(e) N is a geometric random variable with $\frac{5}{6}$ probability of success at each time step. Therefore, $E[N] = \frac{6}{5}$ and $var(N) = \frac{6}{25}$.

(f) To solve this probem we can calculate the expected number of transitions needed to get to state 3 and add the expected number of transitions from state 3 until absorption by $\{5,6\}$. Conditioning on eventually entering the recurrent class $\{5,6\}$ changes the transition probabilities p_{22}, p_{21}, and p_{23}.
Define A as the event that the recurrent class 5,6 is eventually entered.

$$P(X_{n+1} = 3|X_n = 2, A) = \frac{P(X_{n+1} = 3, A|X_n = 2)}{P(A|X_n = 2)}$$

However, since the event that $X_{n+1} = 3$ implies that A is true, the probability can be simplified:

$$\begin{aligned} P(X_{n+1} = 3|X_n = 2, A) &= \frac{P(X_{n+1} = 3, A|X_n = 2)}{P(A|X_n = 2)} \\ &= \frac{P(X_{n+1} = 3|X_n = 2)}{P(A|X_n = 2)} \\ &= \frac{p_{23}}{\frac{p_{23}}{p_{23}+p_{21}}} = p_{23} + p_{21} = \frac{5}{6}. \end{aligned}$$

The self loop probability of state 2 remains the same as can be seen by:

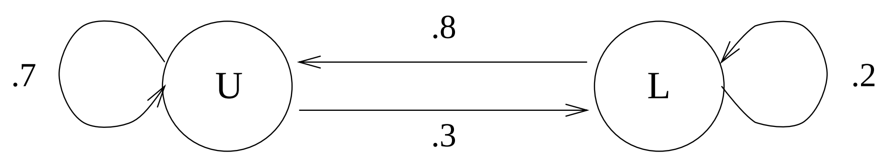
$$\begin{aligned} P(X_{n+1} = 2|X_n = 2, A) &= \frac{P(X_{n+1} = 2, A|X_n = 2)}{P(A|X_n = 2)} \\ &= \frac{P(A|X_{n+1} = 2, X_n = 2)P(X_{n+1} = 2|X_n = 2)}{P(A|X_n = 2)} \\ &= P(X_{n+1} = 2|X_n = 2) = p_{22} = \frac{1}{6}. \end{aligned}$$

Note that the rest of the transition probabilities won't be affected because the recurrent class does not depend on the rest of the chain. Also, the transition probabilities out of the transient states within the chain remain the same.
Therefore, the expected number of transitions to get to state 3 from state 2 is $\frac{6}{5}$.
To calculate the expected number of transitions until absorption by the recurrent class $\{5, 6\}$, we combine $\{5, 6\}$ into one absorbing state and use the expected time to absorption equations.

$$\begin{aligned} \mu_{\{5,6\}} &= 0 \\ \mu_3 &= 1 + \frac{1}{2}\mu_4 + \frac{1}{2}\mu_{\{5,6\}} \\ \mu_4 &= 1 + \frac{3}{4}\mu_3 + \frac{1}{4}\mu_{\{5,6\}} \end{aligned}$$

Therefore, $\mu_3 = \frac{12}{5}$ and $E[M] = \frac{6}{5} + \mu_3 = \frac{18}{5}$.

2. The state-transition diagram is the following:



(a) We are interested in finding the steady-state probabilites of the states in this Markov chain. Since this is a birth-death process, we use the local balance equations based on the frequency of transitions between two successive states and the normalization equation to solve for π_U and π_L.

$$\pi_L = \frac{\pi_U \cdot 3/10}{8/10} = \frac{3}{8}\pi_U$$

$$1 = \pi_L + \pi_U.$$

Solving this system of equations, we get,

$$\pi_U = \frac{8}{11} \qquad \pi_L = \frac{3}{11}.$$

Thus,

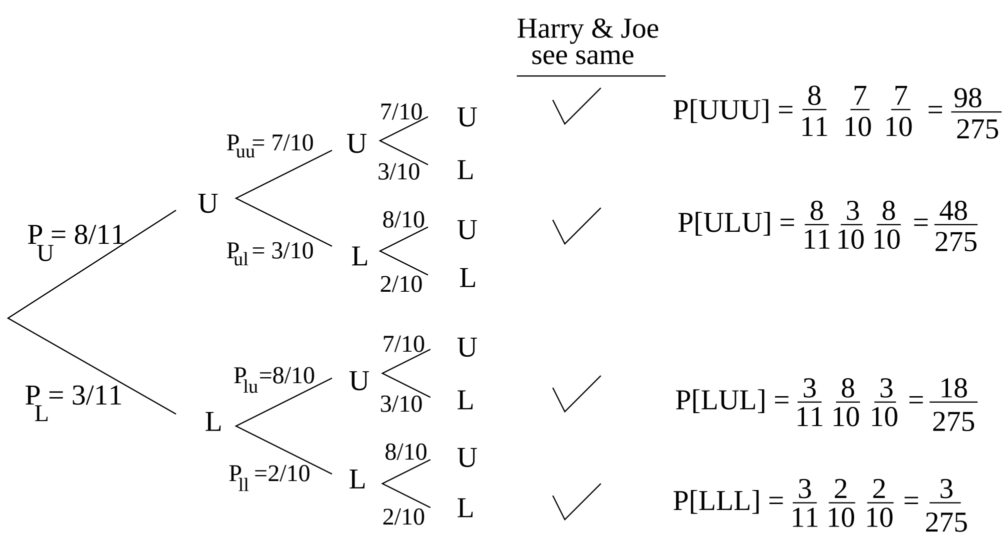
$$\mathbf{P}(\text{he unlocks the door}) = \pi_L \cdot p_{LU} = \frac{3}{11} \cdot \frac{8}{10} = \frac{12}{55}$$

and

$$\mathbf{P}(\text{he locks the door}) = \pi_U \cdot p_{UL} = \frac{8}{11} \cdot \frac{3}{10} = \frac{12}{55}.$$

So, the two events are equally likely.

(b) We can draw a tree of the possible outcomes of Mean Variance's two visits between Joe's arrival and Harry's.



$$\mathbf{P}(\text{both Joe and Harry see the same condition}) = \frac{98}{275} + \frac{48}{275} + \frac{18}{275} + \frac{3}{275} = \boxed{\frac{167}{275}}.$$

(c) Define

X = number of visits from hiring to locking

Y = number of visits from locking to unlocking.

W = number of visits from hiring to unlocking (this is the random variable of interest)

Note that $W = X + Y$. X is a geometric random variable with success probability equal to 0.3 and Y is a geometric random variable with success probability equal to 0.8:

$$p_X(x) = \frac{3}{10}\left(\frac{7}{10}\right)^{x-1}, x = 1, 2, 3, \ldots$$
$$p_Y(y) = \frac{8}{10}\left(\frac{2}{10}\right)^{y-1}, y = 1, 2, 3, \ldots$$

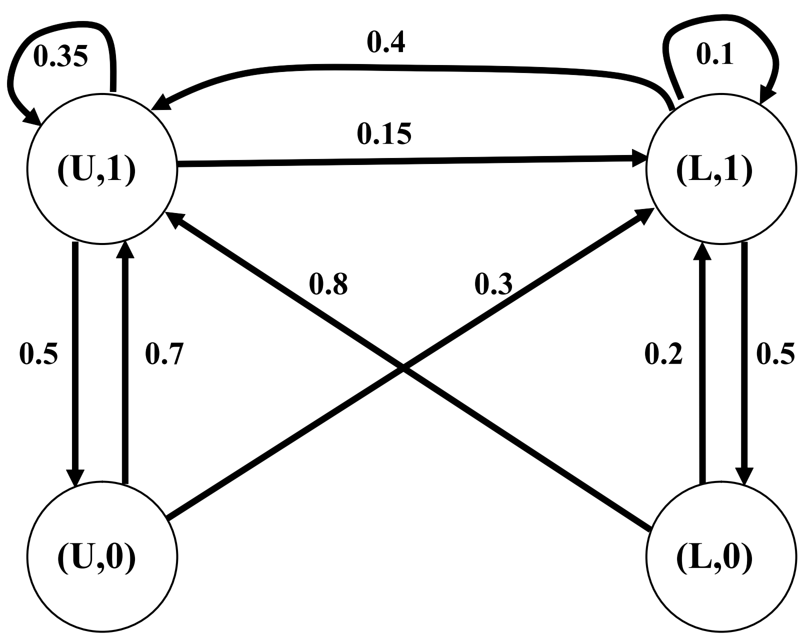
Using the linearity property of expectation and the expected value of a geometric random variable we obtain,

$$\begin{aligned} E[W] &= E[X] + E[Y] \\ &= \frac{10}{3} + \frac{10}{8} \\ &\approx 4.583 \end{aligned}$$

(d) We define a state of two random variables (S, I), where $S \in U, L$ denotes whether the door is unlocked or locked, and $I \in 1, 0$ denotes whether he visits the door at the begining of the hour. Therefore, there are four states in total:

$(U, 1)$ = Door is unlocked and he visits the door at the beginning of the hour
$(U, 0)$ = Door is unlocked and he does not visit the door at the beginning of the hour
$(L, 1)$ = Door is locked and he visits the door at the beginning of the hour
$(L, 0)$ = Door is locked and he does not visit the door at the beginning of the hour

Using the above states, the transition probability graph is given by



For the above Markov chain, the steady state probabilites satisfy the following equations

$$\pi_{U,1} = 0.35\pi_{U,1} + 0.7\pi_{U,0} + 0.4\pi_{L,1} + 0.8\pi_{L,0}$$

$$\begin{aligned}\pi_{U,0} &= 0.5\pi_{U,1}\\ \pi_{L,1} &= 0.15\pi_{U,1} + 0.3\pi_{U,0} + 0.1\pi_{L,1} + 0.2\pi_{L,0}\\ \pi_{L,0} &= 0.5\pi_{L,1}\end{aligned}$$

Moreover, we have the following normalization equation,

$$\pi_{U,1} + \pi_{U,0} + \pi_{L,1} + \pi_{L,0} = 1.$$

Solving these equations, we obtain

$$\begin{aligned}\pi_{U,1} &= \frac{16}{33}\\ \pi_{U,0} &= \frac{8}{33}\\ \pi_{L,1} &= \frac{6}{33}\\ \pi_{L,0} &= \frac{3}{33}\end{aligned}$$

Therefore, the probability of seeing the door unlocked is

$$\mathbf{P}(U) = \pi_{U,1} + \pi_{U,0} = \frac{16}{33} + \frac{8}{33} = \frac{8}{11}.$$

Comparing to Part (a), the answer does not change at all.

(e) As in Part (b), we can draw a tree of the possible states between the two visits. Here, we list all the possible state transitions which satify the requirement that both the door states are the same for the two visits, as follows,

$(U,1)->(U,1)->(U,1)$ with probability of $\frac{16}{33} \times \frac{35}{100} \times \frac{35}{100}$
$(U,1)->(U,1)->(U,0)$ with probability of $\frac{16}{33} \times \frac{35}{100} \times \frac{50}{100}$
$(U,1)->(U,0)->(U,1)$ with probability of $\frac{16}{33} \times \frac{50}{100} \times \frac{70}{100}$
$(U,1)->(L,1)->(U,1)$ with probability of $\frac{16}{33} \times \frac{15}{100} \times \frac{40}{100}$
$(U,0)->(U,1)->(U,1)$ with probability of $\frac{8}{33} \times \frac{70}{100} \times \frac{35}{100}$
$(U,0)->(U,1)->(U,0)$ with probability of $\frac{8}{33} \times \frac{70}{100} \times \frac{50}{100}$
$(U,0)->(L,1)->(U,1)$ with probability of $\frac{8}{33} \times \frac{30}{100} \times \frac{40}{100}$
$(L,1)->(U,1)->(L,1)$ with probability of $\frac{6}{33} \times \frac{40}{100} \times \frac{50}{100}$
$(L,1)->(L,1)->(L,1)$ with probability of $\frac{6}{33} \times \frac{10}{100} \times \frac{10}{100}$
$(L,1)->(L,1)->(L,0)$ with probability of $\frac{6}{33} \times \frac{10}{100} \times \frac{50}{100}$
$(L,1)->(L,0)->(L,1)$ with probability of $\frac{6}{33} \times \frac{50}{100} \times \frac{20}{100}$
$(L,0)->(U,1)->(L,1)$ with probability of $\frac{3}{33} \times \frac{80}{100} \times \frac{50}{100}$
$(L,0)->(L,1)->(L,1)$ with probability of $\frac{3}{33} \times \frac{20}{100} \times \frac{10}{100}$
$(L,0)->(L,1)->(L,0)$ with probability of $\frac{3}{33} \times \frac{20}{100} \times \frac{50}{100}$

The probability with which Joe and Harry see the same condition is equal to the sum of all the probabilites of the above transitions. It follows that

$$\mathbf{P}(\text{both Joe and Harry see the same condition}) = \frac{173}{275},$$

which is larger than the answer in Part (b).