6.041 Fall 2007 Final (Monday, December 17)
DO NOT TURN THIS PAGE OVER UNTIL TOLD

Name: ______________________

Recitation Instructor: ______________________

TA: ______________________

General Instructions:

- **Please take out your MIT ID and place it on your table as it will be checked.**
- This exam has 6 problems. Problem 0 is worth 3 points, Problem 1 is worth 21 points, Problem 2 is worth 13 points and Problem 3 is worth 27 points, Problem 4 is worth 20 points, Problem 5 is worth 16 points, for a total of 100 points. You have 3 hours to complete the exam.
- A standard normal table is available on the last page.
- Write your solutions in this quiz booklet. Only solutions in this quiz booklet will be graded. **Use blue booklets for scratch paper and hand them in. They will be kept but not graded.**
- You are allowed three double sided, handwritten 8.5" by 11" formula sheets plus a calculator.
- **Except where a numerical answer is required**, you may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of numbers that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- Be neat! You will not get credit if we can't read it.
- To receive full credit, for all but Problem 0 and Problem 1, **show your calculations and briefly explain your reasoning so that we can understand your approach.**

Problem 0: (*3 points*) Write your name, your assigned recitation instructor's name, and assigned TA's name on the cover of the quiz booklet. The Instructor/TA pairing is listed below.

Recitation Instructor	TA	Rec. Time
Shivani Agarwal	Danielle Hinton	10 & 11
Franz Kaertner	Jessica Wu	12
Alan Willsky	Miranda Ha	1 & 2
Alex (Sasha) Megretski	Han Wang	1 & 2
Devavrat Shah (6.431)	Faisal Kashif	9:30 & 10:30

Problem	Points	Grade	Grader
0	3		
1 (a)	4		
1 (b)	8		
1 (c)	5		
1 (d)	2		
1 (e)	2		
2 (a)	8		
2 (b)	5		
3 (a)	6		
3 (b)	6		
3 (c)	8		
3 (d)	7		
4 (a)	6		
4 (b)	6		
4 (c)	8		
5 (a)	10		
5 (b)	6		
Total	100		

Problem 1 (*21 points*)

For the following six questions, clearly check ($\surd$) true or false for each statement. For this problem, there is **no need to explain your answers** as **no partial credit will be given**.

(a) (*4 points*) Which statements are true for ALL pairs of random variables X and Y (not necessarily independent)?

(i) $\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y]$
$\underline{\surd}$True ___False

(ii) $\mathbf{E}[YX] = \mathbf{E}[X]\mathbf{E}[Y]$
___True $\underline{\surd}$False

(iii) $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$
___True $\underline{\surd}$False

(iv) $\text{var}(XY) = \text{var}(X)\text{var}(Y)$
___True $\underline{\surd}$False

(b) (*8 points*) Let $\{X_1, X_2, X_3, \ldots\}$ be a Bernoulli process with probability of success equal to p. Which statements must be true?

(i) The probability that the 3rd success occurs on the 7th trial is $\binom{7}{3} p^3(1-p)^4$.
___True $\underline{\surd}$False

(ii) The expected number of failures before the 3rd success is $\frac{3}{p}$.
___True $\underline{\surd}$False

(iii) The variance of the number of failures before the 3rd success is $\frac{3(1-p)}{p^2}$.
$\underline{\surd}$True ___False

(iv) If the given Bernoulli process is merged with a second independent Bernoulli process $\{Y_1, Y_2, Y_3, \ldots\}$ with probability of a success equal to q, the result is a new Bernoulli process with probability of a success equal to $p+q$.
___True $\underline{\surd}$False

(c) (*5 points*) Which of the following expressions are always equal to $\text{var}(X)$?

(i) $2\text{var}(.5X)$
___True $\underline{\surd}$False

(ii) $\mathbf{E}\Big[X^2 - 2X\mathbf{E}[X] + (\mathbf{E}[X])^2\Big]$
$\underline{\surd}$True ___False

(iii) $\mathbf{E}[X^2] - (\mathbf{E}[X])^2$
$\underline{\surd}$True ___False

(iv) $2\text{var}(.5X) + cov(.5X, .5X)$
___True $\underline{\surd}$False

(v) $cov(X, X)$
$\underline{\surd}$True ___False

(d) (*2 points*) If $X \geq 0$, then $\mathbf{P}(X \geq aE[X]) \leq \frac{1}{a}$ for $a > 0$.
$\underline{\surd}$True ___False

(e) (*2 points*) For any two random variables X and Y, $\mathbf{E}[X|Y] = \mathbf{E}[X]$ implies that X and Y are independent.
___True $\underline{\surd}$False

Problem 2 (*13 points*) The number of photons N emitted by a laser in any time interval of length $t = 1$ follows a Poisson distribution

$$p_N(n;\lambda) = e^{-\lambda}\frac{\lambda^n}{n!}, n = 0, 1, 2, 3, \ldots$$

You are given the task of estimating the average photon flux λ from the laser by counting the photons $N_k = n_k$, in non-overlapping time intervals of length $t = 1$. The photon numbers in non-overlapping time intervals are statistically independent.

(a) (*8 points*) Find the maximum likelihood estimator $\widehat{\Lambda} = h(N_1, ..., N_m)$ for the average photon flux λ using the observed photon numbers $N_1 = n_1, ..., N_2 = n_2, ..., N_m = n_m$, measured in m non-overlapping time intervals of length $t = 1$. ► Show your calculations and briefly explain your reasoning so that we can understand your approach. ◄

The joint probability distribution for the random variables $N_1, \ldots, N_m$ is the product of the individual probability distributions, i.e. the likelihood function is

$$f_{\underline{N}}(\underline{N};\lambda) = \prod_{k=1}^{m} e^{-\lambda}\frac{\lambda^{n_k}}{n_k!} = e^{-m\lambda}\prod_{k=1}^{m}\frac{\lambda^{n_k}}{n_k!}$$

and its logarithmic likelihood is

$$\ln f_{\underline{N}}(\underline{N};\lambda) = -m\lambda + \sum_{k=1}^{m} n_k \ln\lambda - \sum_{k=1}^{m}\ln n_k!$$

We look for the parameter that maximize the logarithmic likelihood:

$$\begin{aligned}
\frac{\partial}{\partial\lambda} &= -m + \sum_{k=1}^{m}\frac{n_k}{\hat{\lambda}} = 0 \\
\hat{\lambda} &= \frac{1}{m}\sum_{k=1}^{m} n_k \\
\hat{\Lambda} &= \frac{1}{m}\sum_{k=1}^{m} N_k
\end{aligned}$$

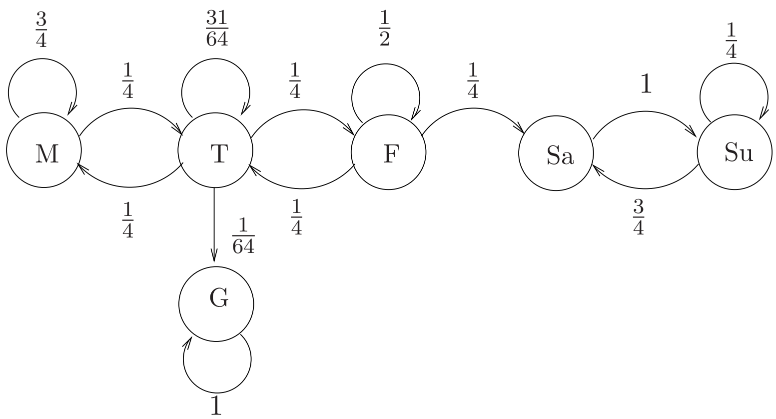
(b) (*5 points*) Is the estimator (i) biased, (ii) asymptotically unbiased, (iii) consistent? ► Show your calculations and briefly explain your reasoning so that we can understand your approach. ◄

$$E[\hat{\lambda}] = \frac{1}{m}\sum_{k=1}^{m} E[N_k] = \frac{1}{m}\sum_{k=1}^{m}\lambda = \lambda$$

Therefore, the estimator is (i) unbiased, (ii) asymptotically unbiased, and (iii) consistent via the WLLN.

Problem 3 (*27 points*)

In a strange land far away, there are 6 kinds of days: Mondin (M), Tuesdin (T), Fridin (F), Saturdin (Sa), Sundin (Su), and Goldin (G). Of these, Sundin and Goldin are holidays; the rest are all working days. In this strange land, days do not necessarily progress sequentially from one kind to another; instead, they progress according to a random process described by the following Markov chain (thus, if today is a Mondin, then with probability 1/4, tomorrow will be a Tuesdin, and with probability 3/4, it will again be a Mondin):

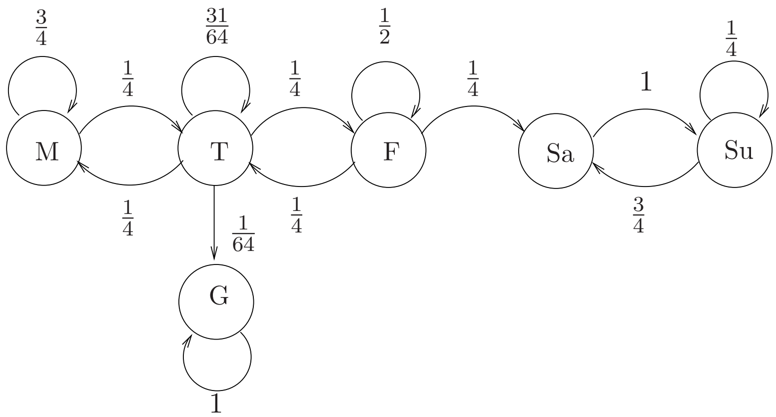


(a) (*6 points*) If today is a Mondin, what is the probability that you will eventually find yourself in a Goldin? (**Half credit for correct setup of equations, half credit for numerical solutions.**) ► Show your calculations and briefly explain your reasoning so that we can understand your approach. ◄

We are interested in finding a_M, the probability that starting from state $i = M$, we will eventually reach state $s = G$. Using the system of equations for absorption probabilities, we have

$$\begin{aligned}
a_M &= p_{MM}a_M + p_{MT}a_T &&= \tfrac{3}{4}a_M + \tfrac{1}{4}a_T \\
a_T &= p_{TM}a_M + p_{TT}a_T + p_{TF}a_F + p_{TF}a_G &&= \tfrac{1}{4}a_M + \tfrac{31}{64}a_T + \tfrac{1}{4}a_F + \tfrac{1}{64}a_G \\
a_F &= p_{FT}a_T + p_{FF}a_F &&= \tfrac{1}{4}a_T + \tfrac{1}{2}a_F \\
a_G &= 1
\end{aligned}$$

Solving, from the first equation $a_M = a_T$. From the third equation, $a_F = \frac{1}{2}a_T$. Then from the second equation $\left(1 - \frac{31}{64} - \frac{1}{4}\frac{1}{2}\right) a_T = \frac{1}{4}a_M + \frac{1}{64} \Rightarrow a_T = \frac{16}{25}a_M + \frac{1}{25}$. Combining the results, $\frac{9}{25}a_M = \frac{1}{25}$, or $a_M = \frac{1}{9}$.



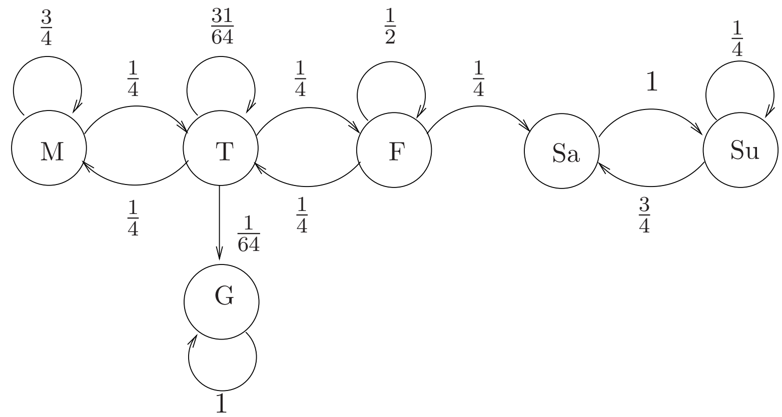
Repeated figure of Markov chain

(b) (*6 points*) If today is a Fridin, what is the expected number of days you will have to wait for the next holiday? (**Half credit for correct setup of equations, half credit for numerical solutions.**) ▶ Show your calculations and briefly explain your reasoning so that we can understand your approach. ◀

Using the system of equations for the expected time to absorption (where in this problem, we take absorption states as those states corresponding to a holiday), we have

$$\begin{aligned}
\mu_M &= 1 + p_{MM}\mu_M + p_{MT}\mu_T &&= 1 + \tfrac{3}{4}\mu_M + \tfrac{1}{4}\mu_T \\
\mu_T &= 1 + p_{TM}\mu_M + p_{TT}\mu_T + p_{TF}\mu_F + p_{TG}\mu_G &&= 1 + \tfrac{1}{4}\mu_M + \tfrac{31}{64}\mu_T + \tfrac{1}{4}\mu_F + \tfrac{1}{64}\mu_G \\
\mu_F &= 1 + p_{FT}\mu_T + p_{FF}\mu_F + p_{FSa}\mu_{Sa} &&= 1 + \tfrac{1}{4}\mu_T + \tfrac{1}{2}\mu_F + \tfrac{1}{4}\mu_{Sa} \\
\mu_G &= 0 \\
\mu_{Sa} &= 1 \\
\mu_{Su} &= 0
\end{aligned}$$

Solving, we find $\mu_M = \frac{68}{3}$, $\mu_T = \frac{56}{3}$, $\mu_F = \frac{71}{6}$. Thus, the answer is $\mu_F = \frac{71}{6}$.



Repeated figure of Markov chain

(c) (*8 points*) Every Saturdin marks the beginning of a new weekin, which is defined as a Saturdin and all the ***consecutive*** Sundins that follow it. Every Saturdin you earn \$2.00, and on every Sundin you spend \$1.00. You have agreed to report to your spouse your net savings S (earnings minus expenses) over the next 90 weekins. (Note: this 90 weekin period begins with a Saturdin and ends with a Sundin.) Find the expectation and variance of S. ► Show your calculations and briefly explain your reasoning so that we can understand your approach. ◄

Let $S = \sum_{i=1}^{90} X_i$ where X_i is the winning in each weekin.

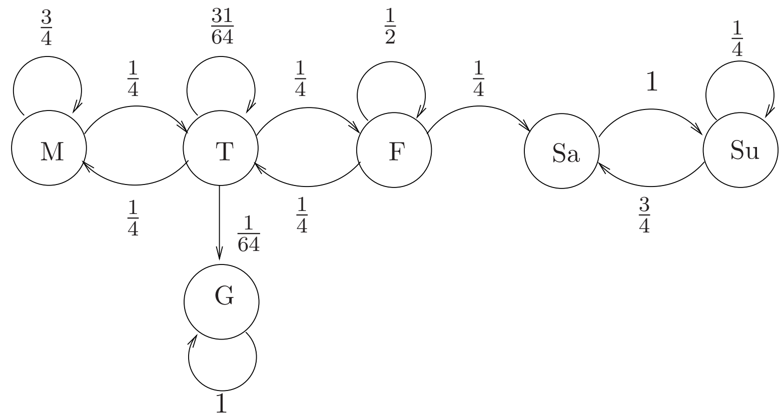
For each weekin, there is one Saturdin and the number of Sundins is a geometric random variable with $p = \frac{3}{4}$. So $E[X_i] = 2 - \frac{4}{3} \cdot 1 = \frac{2}{3}$ and $var(X_i) = \frac{1-\frac{3}{4}}{(\frac{3}{4})^2} = \frac{4}{9}$.

Hence,

$$E[S] = 90 \cdot E[X_i] = 90 \cdot \frac{2}{3} = 60$$

and

$$var(S) = 90 \cdot var(X_i) = 90 \cdot \frac{4}{9} = 40$$



Repeated figure of Markov chain

(d) (*7 points*) Find a good numerical approximation to the probability your savings S from part (c) will be at least \$50.00. At the end of your calculations, the answer should take the form of a numerical value for $\mathbf{P}(S \geq \$50.00)$. ► Show your calculations and briefly explain your reasoning so that we can understand your approach. ◄

From the Central Limit Theorem (because S is a sum of a large number of independent random variables), we can approximate S with a normal distribution.

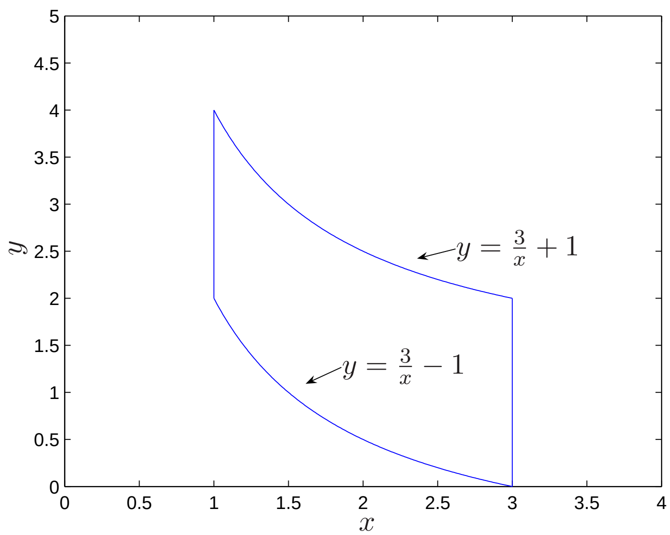
$$\begin{aligned}
P(S \geq \$50) &= P\left(\mathcal{N}(0,1) \geq \frac{50-60}{\sqrt{40}}\right) \\
&= P\left(\mathcal{N}(0,1) \geq -1.58\right) \\
&= P\left(\mathcal{N}(0,1) < 1.58\right) \\
&= 0.9429
\end{aligned}$$

Problem 4 (*20 points*) In this problem we will estimate X, given a noisy measurement Y, where

$$Y = \frac{3}{X} + N,$$

with X uniformly distributed in [1,3], the noise N uniformly distributed in [-1,1], and X and N independent.

(a) (*6 points*) Find the conditional density $f_{Y|X}(y|x)$ and plot the region in the x, y plane where it is nonzero. Identify the boundaries of the region clearly. ► Show your calculations and briefly explain your reasoning so that we can understand your approach. ◄



Since $Y = \frac{3}{X} + N$, the joint distribution $f_{X,Y}(x,y)$ is uniform in the region shown. Hence, $f_{Y|X}(y|x)$ is uniform distribution in the range $[3/x - 1, 3/x + 1]$.

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{2}, & \text{for} \quad 1 \le x \le 3, \frac{3}{x} - 1 \le y \le \frac{3}{x} + 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) (*6 points*) Find the joint density $f_{X,Y}(x,y)$. In what region is your formula valid? ► Show your calculations and briefly explain your reasoning so that we can understand your approach. ◄

By definition, $f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x)$. Since $f_X(x)$ is uniform in [1,3] and we have $f_{Y|X}(y|x)$ from part(a), then

$$f_{Y,X}(y,x) = \begin{cases} \frac{1}{4}, & \text{for} \quad 1 \le x \le 3, \frac{3}{x} - 1 \le y \le \frac{3}{x} + 1 \\ 0, & \text{otherwise} \end{cases}$$

(c) (*8 points*) Find the minimum mean-squared error estimator $\hat{X}_{MMSE} = g(Y)$ for X, based on Y. (You can do this without explicitly finding $f_Y(y)$ or $f_{X|Y}(x|y)$). ► Show your calculations and briefly explain your reasoning so that we can understand your approach. ◄

$X_{\hat{MMSE}} = g(Y) = E[X|Y]$ For $0 \le y \le 2$, $f_{X|Y}(x|y)$ is uniform in the range $[\frac{3}{y+1}, 3]$, and for $2 \le y \le 4$, $f_{X|Y}(x|y)$ is uniform in the range $[1, \frac{3}{y-1}]$. Hence,

$$E[X|Y] = \begin{cases} \frac{1+\frac{3}{y-1}}{2} = \frac{y+2}{2(y-1)}, & \text{for} \quad 2 \le y \le 4 \\ \frac{3+\frac{3}{y+1}}{2} = \frac{3y+6}{2(y+1)}, & \text{for} \quad 0 \le y \le 2 \\ \text{undefined}, & \text{for} \quad y < 0 \text{ or } y > 4 \end{cases}$$

Problem 5 (*16 points*) A carnival magician has 3 coins that all look the same. Two are ordinary and have $\mathbf{P}(heads) = \mathbf{P}(tails) = p = \frac{1}{2}$. One is special and has $\mathbf{P}(heads) = q = 0.7$, $\mathbf{P}(tails) = 1-q = 0.3$. In every game a customer is allowed to pick one of the coins at random, flip it 3 times, and then guess whether it was the special coin or an ordinary one. The coin is then illuminated with an ultraviolet lamp, under which the special coin fluoresces a bright white while the ordinary coins remain dark. The player wins \$1.00 if his or her guess was correct. The player wins nothing for an incorrect guess.

(a) (*10 points*) What is your best strategy for using the information from all three previous tosses to identify the coin with the minimum probability of error? ▶ Show your calculations and briefly explain your reasoning so that we can understand your approach. ◀

Let

H_0 be the event that you have picked an ordinary ($p = 1/2$) coin
H_1 be the event that you have picked a special ($p = .3$) coin
X be the number of tails in the three tosses.

Then the likelihood ratio is

$$L(x) = \frac{P[X|H_1]}{P[X|H_0]} \gtrless \frac{P[H_0]}{P[H_1]}$$

$$\frac{\binom{3}{x}(0.7)^{(3-x)}(0.3)^x}{\binom{3}{x}(0.5)^3} \gtrless \frac{\frac{2}{3}}{\frac{1}{3}}$$

$$(0.7)^{(3-x)}(0.3)^x \gtrless \frac{1}{4}$$

With numbers

$$(0.7)^{(3-x)}(0.3)^x = \begin{cases} 0.343, & x = 0 \\ 0.147, & x = 1 \\ 0.063, & x = 2 \\ 0.027, & x = 3 \end{cases}$$

Thus,

$$\begin{aligned} X = 0 : &\qquad \text{guess } H_1 \text{ (special coin)} \\ X = 1, 2, 3 : &\qquad \text{guess } H_0 \text{ (ordinary coin)} \end{aligned}$$

(b) (*6 points*) What is your expected amount won per game using the strategy in part (a)? ▶ Show your calculations and briefly explain your reasoning so that we can understand your approach. ◀

We first find the probability of error.

$$p_e = P(g(X) = H_1|H_0)P(H_0) + P(g(X) = H_0|H_1)P(H_1) = \alpha P(H_0) + \beta P(H_1)$$

With

$$\begin{aligned} \alpha &= p_{FA} = P(g(X) = H_1|H_0) = P\left(X = 0|H_0\right) = \binom{3}{0}\left(\frac{1}{2}\right)^3 = \frac{1}{8} \\ \beta &= p_M = P(g(X) = H_0|H_1) = P(X = 1, 2, 3|H_1) = 1 - \binom{3}{0}(0.7)^3(0.3)^0 \approx 0.657 \\ p_e &= (0.125)\left(\frac{2}{3}\right) + (0.657)\left(\frac{1}{3}\right) = 0.302 \end{aligned}$$

The expected winning per game is $1 - p_e = \$0.697$.

The standard normal table.

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	**0.7939**	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Cumulative Distribution Function for the standard Gaussian ($\mathcal{N}(0,1)$) random variable. For example the shaded entry, **0.7939**, equals

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.82} e^{-x^2/2} dx.$$