Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2004)

Final Exam Solutions

Problem 1

1.1 Let R and S denote the number of regular and spam messages received, respectively. Let T_1, T_2, ... be the processing times for the regular messages and T be the total processing time, all in seconds. Then

$$T = 2S + T_1 + T_2 + \cdots + T_R,$$

and the question asks for $\mathbf{E}[T]$ and $\text{var}(T)$.

Using elementary properties of sums of independent random variables and the standard formulas for sums of random numbers of independent random variables gives

$$\begin{aligned}
\mathbf{E}[T] &= 2\mathbf{E}[S] + \mathbf{E}[T_1 + T_2 + \cdots T_R] \qquad \text{(elementary)} \\
&= 2\mathbf{E}[S] + \mathbf{E}[T]\,\mathbf{E}[R] \qquad \text{(formula for random sums)} \\
&= 2 \cdot 80 + 90 \cdot 20 \;=\; 1960
\end{aligned}$$

and

$$\begin{aligned}
\text{var}(T) &= 4\text{var}(S) + \text{var}(T_1 + T_2 + \cdots + T_R) \qquad \text{(elementary)} \\
&= 4\text{var}(S) + \text{var}(T)\mathbf{E}[R] + (\mathbf{E}[T])^2\text{var}(R) \qquad \text{(formula for random sums)} \\
&= 4 \cdot 80 + \frac{(120-60)^2}{12} \cdot 20 + 90^2 \cdot 20 \;=\; 168320
\end{aligned}$$

1.2 When we consider regular and spam email together, we have a Poisson arrival process at rate 10 messages per hour where each arrival has probability $\frac{2}{2+8} = \frac{1}{5}$ of being a regular message.

$$\mathbf{P}(\text{party invitation}) = \mathbf{P}(\text{regular mail}) \cdot \mathbf{P}(\text{party invitation} \mid \text{regular mail}) = \frac{1}{5} \cdot \frac{1}{20} = \frac{1}{100}.$$

1.3 Party invitations arrive as a Poisson process with rate $\lambda_p = 0.1$ messages per hour. The number of party invitations to arrive in a 10-hour period is thus a Poisson random variable with parameter $0.1 \cdot 10 = 1$. The desired PMF is

$$p(k) = \begin{cases} e^{-1}\frac{1^k}{k!}, & k = 0, 1, 2, \ldots; \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{e}\frac{1}{k!}, & k = 0, 1, 2, \ldots; \\ 0, & \text{otherwise.} \end{cases}$$

1.4 The key thing here is consider the process consisting of party invites, regular emails which are not party invites, and spam altogether. Given there is at least one party invite, then the probability of getting ≥ 1 email is 1. If there are 0 party invites, then the probability of getting at least one email is just the probability of an arrival in the combined process of spam and regular emails which are not party invites.

$$\begin{aligned}
\mathbf{P}(k \text{ party invites} \mid \geq 1 \text{ email}) &= \frac{\mathbf{P}(\geq 1 \text{ email} \mid k \text{ party invites})\,\mathbf{P}(k \text{ party invites})}{\mathbf{P}(\geq 1 \text{ emails})} \\
&= \frac{\mathbf{P}(\geq 1 \text{ email} \mid k \text{ party invites})\,\frac{1}{e}\frac{1}{k!}}{1 - e^{-100}} \\
&= \begin{cases} \frac{(1-e^{-99})\frac{1}{e}}{1-e^{-100}}, & k = 0 \\ \frac{\frac{1}{e}\frac{1}{k!}}{1-e^{-100}}, & k = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

1.5 Let S be the number of spam messages received out of the first 100 total messages. S is a binomial(100,$\frac{4}{5}$) random variable so

$$\mathbf{P}(S=80) = \binom{100}{80}\left(\frac{4}{5}\right)^{80}\left(\frac{1}{5}\right)^{20}.$$

As a precursor to the De Moivre–Laplace approximation (based on the Central Limit Theorem), we compute $\mathbf{E}[S] = 100 \cdot \frac{4}{5} = 80$ and $\text{var}(S) = 100 \cdot \frac{4}{5} \cdot \frac{1}{5} = 16$. Now we can compute

$$\begin{aligned}
\mathbf{P}(S=80) &= \mathbf{P}(79.5 \le S \le 80.5) \\
&= \mathbf{P}\left(\frac{79.5-80}{4} \le \frac{S-80}{4} \le \frac{80.5-80}{4}\right) \\
&= \mathbf{P}\left(-0.125 \le \frac{S-80}{4} \le 0.125\right) \\
&\approx \Phi(0.125) - \Phi(-0.125) \qquad \text{since } (S-80)/4 \text{ is approximately standard normal} \\
&= 2\Phi(0.125) - 1 \\
&\approx 2 \cdot 0.5497 - 1 \;=\; 0.0995
\end{aligned}$$

1.6 Let T be the time asleep (in hours) and M be the number of messages that arrive while asleep. We are given that T is an exponential(1) random variable. Given $T = t$, M is a Poisson random variable with parameter $10t$. Now a version of Bayes's Rule gives (for $t \ge 0$)

$$\begin{aligned}
f_{T|\{M=3\}}(t) &= \frac{p_{M|T}(3 \mid t) f_T(t)}{p_M(3)} = \frac{p_{M|T}(3 \mid t) f_T(t)}{\int_0^\infty p_{M|T}(3 \mid t) f_T(t)\, dt} \\
&= \frac{\frac{e^{-2t}(2t)^3}{3!} e^{-t}}{\int_0^\infty \frac{e^{-2t}(2t)^3}{3!} e^{-t}\, dt} = \frac{3^4 t^3 e^{-3t}}{3!}
\end{aligned}$$

There are a couple of ways to get to the final answer. One is to just do the integration (by parts). Another is to consider the probability $P_M(3)$. Think of a combined process of regular email arrivals and "sleep awakening" arrivals. Then this process is Poisson with parameter 3 (2 for emails, 1 for sleep), and $P_M(3)$ is the probability that three email arrivals occur before a sleep arrival. A third way is to recognize that the numerator has the form of an Erlang of order 4 with parameter 3. Then the denominator is the constant necessary to normalize the pdf. $P_M(3) = 8/81$.

To give a complete answer, we should also note that

$$f_{T|\{M=3\}}(t) = 0 \qquad \text{for } t < 0.$$

Problem 2

2.1 All three states are recurrent. There is a single recurrent class which is aperiodic.

2.2 We need three linearly independent equations involving unknowns π_1, π_2, and π_3. The standard technique for this is to have the normalization equation and any two balance equations, for example:

$$\begin{aligned} \pi_1 + \pi_2 + \pi_3 &= 1 \\ -\frac{1}{2}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3 &= 0 \\ \frac{1}{3}\pi_2 - \frac{1}{2}\pi_3 &= 0 \end{aligned}$$

Solving gives $\pi_1 = \frac{4}{9}$, $\pi_2 = \frac{1}{3}$, $\pi_3 = \frac{2}{9}$.

2.3 Define t_i as the expected time to first enter state 3 starting from state i, $i = 1, 2, 3$. By using the total expectation theorem with conditioning on the possible next states from any given state, one obtains:

$$\begin{aligned} t_1 &= 1 + \frac{1}{2}t_1 + \frac{1}{2}t_2 \\ t_2 &= 1 + \frac{1}{3}t_1 + \frac{1}{3}t_2 + \frac{1}{3}t_3 \\ t_3 &= 0 \end{aligned}$$

Solving gives $t_1 = 7$, $t_2 = 5$, $t_3 = 0$. The final answer is now obtained with the total expectation theorem by conditioning on the possible initial states:

$$\begin{aligned} \mathbf{E}\left[\text{time to first enter state 3}\right] &= \mathbf{E}\left[\text{time to first enter state 3} \mid X_0 = 1\right] \mathbf{P}\left(X_0 = 1\right) \\ &\quad + \mathbf{E}\left[\text{time to first enter state 3} \mid X_0 = 2\right] \mathbf{P}\left(X_0 = 2\right) \\ &= t_1 \cdot \frac{1}{3} + t_2 \cdot \frac{2}{3} = \frac{17}{3}. \end{aligned}$$

2.4 We can find $\mathbf{E}\left[T_i\right]$ using the t_is computed above and the total expectation theorem (where the conditioning is on the three possible transitions from state 3):

$$\begin{aligned} \mathbf{E}\left[T_i\right] &= \mathbf{E}\left[\text{time to return to 3} \mid \text{next state is 1}\right] \mathbf{P}\left(\text{next state is 1}\right) \\ &\quad + \mathbf{E}\left[\text{time to return to 3} \mid \text{next state is 2}\right] \mathbf{P}\left(\text{next state is 2}\right) \\ &\quad + \mathbf{E}\left[\text{time to return to 3} \mid \text{next state is 3}\right] \mathbf{P}\left(\text{next state is 3}\right) \\ &= (1 + t_1) \cdot \frac{1}{2} + t_2 \cdot 0 + 1 \cdot \frac{1}{2} \\ &= \frac{9}{2}. \end{aligned}$$

2.5 Because of the Markov property, T_1, T_2, ... are identically distributed. Thus there can only be convergence in probability if there is some constant a such that $\mathbf{P}\left(T_1 = a\right) = 1$. This is obviously not the case because there are so many distinct state sequences that start in 3 and end when they return to 3 for the first time (3123, 312123, 31212123, ...).

2.6 Since the T_is are identically distributed and $\mathbf{E}[T_i]$ is finite, the convergence in probability of Q_n follows from the weak law of large numbers. This convergence in probability is to the value $\mu = \mathbf{E}[T_i]$.

2.7 Conditioned on $X_0 = 1$, Y is a normal random variable with mean 1 and variance 1.

Let $Z = W^2$ where W is a normal random variable with mean 1 and variance 1. The standard approach to finding the PDF of Z is to first find the CDF and then differentiate:

$$\begin{aligned} F_Z(z) &= \mathbf{P}(Z \le z) \\ &= \mathbf{P}\left(W^2 \le z\right) \\ &= \mathbf{P}\left(-\sqrt{z} \le W \le \sqrt{z}\right) \qquad \text{for } z > 0 \text{ (otherwise 0)} \\ &= F_W(\sqrt{z}) - F_W(-\sqrt{z}) \qquad \text{for } z > 0 \text{ (otherwise 0)} \end{aligned}$$

$$\begin{aligned} f_Z(z) &= \frac{d}{dz} F_Z(z) \\ &= \frac{1}{2\sqrt{z}} f_W(\sqrt{z}) + \frac{1}{2\sqrt{z}} f_W(-\sqrt{z}) \qquad \text{for } z > 0 \text{ (otherwise 0)} \\ &= \frac{1}{2\sqrt{z}} \left(\frac{1}{\sqrt{2\pi}} e^{-(\sqrt{z}-1)^2/2} + \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{z}-1)^2/2} \right) \qquad \text{for } z > 0 \text{ (otherwise 0)} \\ &= \frac{1}{2\sqrt{2\pi z}} \left(e^{-(\sqrt{z}-1)^2/2} + e^{-(\sqrt{z}+1)^2/2} \right) \qquad \text{for } z > 0 \text{ (otherwise 0)} \end{aligned}$$

Finally,

$$f_{Y^2|\{X_0=1\}}(z) = f_Z(z) = \begin{cases} \frac{1}{2\sqrt{2\pi z}} \left(e^{-(\sqrt{z}-1)^2/2} + e^{-(\sqrt{z}+1)^2/2} \right), & \text{for } z > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Problem 3 There are two ways to do this problem. The first one is to argue from a Poisson process point of view. X_1 is the time of the first arrival and $X_1 + X_2$ is the time of the second arrival. So this question is really asking for the distribution of the first arrival time, given that one arrival occurred in a given time span (time for two arrivals to happen). This, we know is uniformly distributed. Hence, $X_1/(X_1 + X_2)$ is uniform on $[0, 1]$. The brute force way also works:

$$
\begin{aligned}
\mathbf{P}\left(\frac{X_1}{X_1+X_2} \le z\right) &= \mathbf{P}\left(X_2 \ge \frac{1-z}{z}X_1\right) \\
&= \int_0^\infty \frac{1}{\pi}e^{-\frac{1}{\pi}y}dy \int_0^{\frac{zy}{1-z}} \frac{1}{\pi}e^{-\frac{1}{\pi}x}dx \\
&= \int_0^\infty \frac{1}{\pi}e^{-\frac{1}{\pi}y}\left(1 - e^{-\frac{1}{\pi}\frac{z}{1-z}y}\right)dy \\
&= 1 - \int_0^\infty \frac{1}{\pi}e^{-\frac{1}{\pi}\frac{y}{1-z}}dy \\
&= 1 - (1-z) \\
&= z
\end{aligned}
$$

Hence, the pdf is uniform on $[0, 1]$.

Problem 4

4.1 False. Consider the sequence of random variables defined as follows:

$$X_n = \begin{cases} 0 & \text{wp } 1 - \frac{1}{n} \\ n & \text{wp } \frac{1}{n} \end{cases}$$

Note that $\mathbf{E}[X_n] = 1$ for all n, but $X_n \to 0$ in probability.

4.2 False.

$$\begin{aligned} \mathbf{P}(\min\{X, Y\} \leq z) &= 1 - \mathbf{P}(X > z, Y > z) \\ &= 1 - e^{-\lambda z} e^{-\mu z} \\ &= 1 - e^{-(\lambda+\mu)z} \end{aligned}$$

Hence, $\min\{X, Y\}$ is an exponential rv with parameter $\lambda + \mu$.

Another way to think about this is by considering a merged Poisson process. The minimum of X and Y is the interarrival time of this merged process, which means that arrivals are coming at a rate of $\lambda + \mu$.

4.3 True.

$$\begin{aligned} cov(X, XY) &= \mathbf{E}\left[X^2 Y\right] - \mathbf{E}[X]\,\mathbf{E}[XY] \\ &= \mathbf{E}\left[X^2\right]\mathbf{E}[Y] - \mathbf{E}[X]\,\mathbf{E}[XY] = 0 \end{aligned}$$

since both X and Y are zero-mean random variables.

Clearly X is Gaussian. For XY,

$$\begin{aligned} \mathbf{P}(XY \leq z) &= \mathbf{P}(XY \leq z | Y = 1)\,\mathbf{P}(Y = 1) + \mathbf{P}(XY \leq z | Y = -1)\,\mathbf{P}(Y = -1) \\ &= \frac{1}{2}[\mathbf{P}(X \leq z) + \mathbf{P}(X \geq -z)] \\ &= \mathbf{P}(X \leq z) \end{aligned}$$

where the last equality comes from the fact that X is symmetric about 0. Hence, XY is a zero-mean Gaussian as well. However, X and XY are clearly not independent since knowing that $X = x$ will tell us that XY is either x or $-x$. That is, $|XY| = |X|$.