6.431 Fall 2005 Final Exam
Wednesday, Dec. 21, 1:30-4:30 p.m.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Name: ______________________

Recitation Instructor: ______________________

TA: ______________________

Question	Score	Out of
1		50
2		50
Your Grade		100

- You have 180 minutes to complete the quiz.
- There are often many different ways to solve any particular problem. Considering the alternatives and choosing the most efficient solution will save your time and effort.
- **Write your solutions in this booklet. We will not consider any work not in this booklet.**
- This test has two problems that are not necessarily in order of difficulty.
- Unless a numerical answer is requested, you may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of <u>numbers</u> that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- This is a closed-book exam except for three double-sided, handwritten, 8.5 by 11 formula sheets.
- Calculators are not allowed.
- Be neat! If we can't read it, we can't grade it.

Problem 1: *(50 points)*

Everyone is invited to the annual Probability Party at M.I.T. The doors open at 7pm. Everyone who arrives before 7pm is sent away. Everyone who arrives after 7pm stays at least until midnight. Undergraduate students arrive to the party according to a Poisson process with rate λ_u students per hour. Graduate students arrive according to a Poisson process with rate λ_g students per hour, independently of the undergraduate students. Independently of everything else, each student arriving at the party is a female with probability p.

a. *(5 points)* Let A be the event that there are exactly 4 students at the party at 9pm. What is the probability of event A?

$\mathbf{P}(A) =$

b. *(5 points)* Let B be the event that out of the first 10 students in the party, exactly 4 are graduate students. What is the probability of event B?

$\mathbf{P}(B) =$

c. *(6 points)* Let W be the time of arrival of the 7th woman to the party. Find the expectation, the variance and the transform of W.

$\mathbf{E}[W] =$

$Var(W) =$

$M_W(s) =$

d. *(6 points)* Let C be the event that the 3rd undergraduate student arrives before the 2nd graduate student. What is the probability of event C?

$\mathbf{P}(C) =$

e. *(7 points)* At 1am no one else is allowed into the party, and there are only 5 remaining guests. All guests depart independently of each other. Each guest's departure time is an exponential random variable with rate μ. What is the expected time until the first of these guests leaves? What is the expected time until the last of the guests leaves?

E[Time until first departure] =

E[Time until last departure] =

f. *(7 points)* Alice leaves the party at some random time after midnight and goes to the M.I.T. bus stop to catch the ride home. Her route is served by one bus that runs 24 hours a day. The time between any two consecutive times when the bus arrives at the M.I.T. stop is distributed uniformly between 40min and 60min, independently of what happens on all other runs. Let L be the time from the last bus arrival at the M.I.T bus stop to the time when Alice gets to the bus stop. Find the expectation of L.

$\mathbf{E}[L] =$

g. *(7 points)* Student ages are distributed exponentially with parameter λ_a. The gate keeper at the party asks every guest how old they are, but the students always exaggerate their age. Independently of his/her true age, any particular student adds an amount that is exponentially distributed with parameter λ_d when answering the gate keeper's questions. Let the true age of a particular student be A, and the age this student told the gate keeper be X. Find the *linear* least squares estimator of A from X.

$\hat{A}_{LLSE}(X)$:

h. *(7 points)* In this question, assume $\lambda_a = \lambda_d$. Find the *best* least squares estimator of A from X. Explain how it compares with your answer in the previous question.

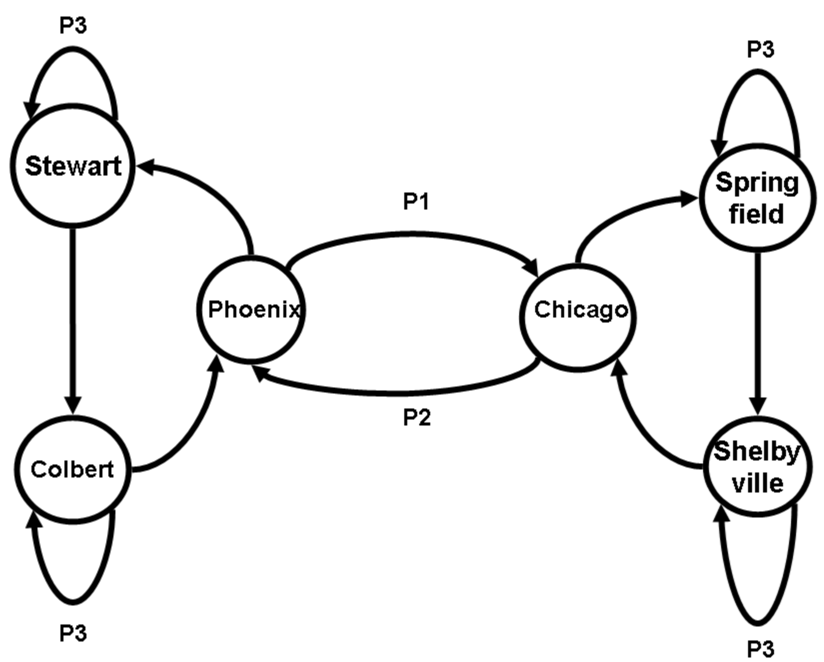
$\hat{A}_{LSE}(X)$:

Compare with your answer from (g), explain:

Problem 2: *(50 points)*

Bernoulli Airline services six airports with a single airplane. The plane makes exactly one flight every day. All flights take place in the middle of the day. The plane schedule can be described by a Markov process shown in the figure below, where p_1, p_2 and p_3 denote transition probabilities. The plane flies among two large cities, Phoenix and Chicago, and four small towns. When the plane is in Phoenix, sometimes it will fly back to Chicago but otherwise it will fly from Phoenix to the small town of Stewart, and then from Stewart to the small town of Colbert, and then finally back to Phoenix. When the plane is in Chicago, sometimes it will fly back to Phoenix, but otherwise it will fly from Chicago to the small town of Springfield, and then from Springfield it will fly to the small town of Shelbyville, and then back to Chicago.

The plane can fly from the large cities regardless of the weather, but it cannot take off from the small airports when the weather is bad. If the weather is bad and the plane is in the small town, it is stuck in that town until the weather is nice again and it can take off. The chance of bad weather on any particular day is independent of what happened on any other day and is equal to p_3.



a. *(5 points)* In this question, assume that the plane starts in Phoenix on Monday morning. Let A be the event that it will be in Stewart on Thursday morning. Find the probability of A.

$\mathbf{P}(A) =$

b. *(5 points)* Suppose the plane is now in the town of Stewart. Let N be the number of days it will take for the plane to arrive in Colbert for the first time from now. Determine the PMF of N.

$p_N(n) =$

c. *(6 points)* Suppose the plane is now in the town of Stewart. Let K be the number of days it will take for the plane to arrive in Phoenix for the first time from now. Determine the PMF of K.

$p_K(k) =$

d. *(6 points)* The airline has been running this schedule for a long time. If the plane is in the air between Phoenix and Chicago, what is the probability that it is headed towards Chicago?

$\mathbf{P}$(Plane is headed to Chicago) =

e. *(7 points)* In this question, assume that global warming suddenly makes the weather so bad that planes can land in small airports, but can no longer take off from small airports. The plane starts either in Chicago or in Phoenix, with equal probability. Find the probability that the plane started in Phoenix given it is now stuck in Stewart.

$\mathbf{P}$(The plane started in Phoenix given it is now stuck in Stewart) =

f. *(7 points)* In this question, again assume that global warming suddenly makes the weather so bad that planes can land in small airports, but can no longer take off from small airports. Also assume that the plane started in Chicago. Let M be the number of flights the plane made before it got stuck in a small town. Find the expectation of M.

$\mathbf{E}[M] =$

g. *(7 points)* The airline has been running this schedule for a long time. It is known that on any given day, the plane is equally likely to be found in any of the six airports. Find the value of p_1, where $0.4 \leq p_1 \leq 0.6$, that maximizes the airline efficiency, that is, maximizes the fraction of the days the plane makes trip on average. Compute also this maximum efficiency.

Your Answer:

h. *(7 points)* In this question, assume that the plane starts in Phoenix and that $p_1 = p_2 = p_3 = \frac{1}{2}$. Let C be the event that by the time the plane takes off from the Stewart airport for the 50th time, it will have spent more than 90 nights at the Stewart airport. Find an *excellent* approximation to the probability of C.

You might find one of the following useful: $\sqrt{10} \approx 3.1623$, $\sqrt{100} = 10$.

$\mathbf{P}(C) \approx$