6.041/6.431 Spring 2008 Quiz 1
Wednesday, March 12, 7:30 - 9:30 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Name: ____________________________

Recitation Instructor: ____________________________

TA: ____________________________

6.041/6.431: ____________________________

Question	Part	Score	Out of
0			3
1	all		40
2	a		5
	b		5
	c		6
	d		6
3	a		5
	b		6
	c		6
	d		6
	e		6
	f		6
	g		10
6.041 Total			100
6.431 Total			110

- Write your solutions in this quiz packet, only solutions in the quiz packet will be graded.
- Question one, multiple choice questions, will receive no partial credit. Partial credit for question two and three will be awarded.
- You are allowed one two-sided 8.5 by 11 formula sheet plus a calculator.
- You have 120 minutes to complete the quiz.
- Be neat! You will not get credit if we can't read it.
- Graded quizzes will be returned in recitation on Tuesday 3/18.

Problem 0: (3 pts) Write your name, your assigned recitation instructor's name, and assigned TA's name on the cover of the quiz booklet. The Instructor/TA pairing is listed below. Also write the class you are registered for: 6.041 or 6.431.

Recitation Instructor	TA	Recitation Time
Vivek Goyal	Natasa Blitvic	10 & 11 AM
Michael Collins	Danielle Hinton	10 & 11 AM
Shivani Agarwal	Stavros Valavanis	12 & 1 PM
Dimitri Bertsekas (6.431)	Aman Chawla (6.431)	1 & 2 PM

Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2008)

Question 1
Multiple Choice Questions: **CLEARLY** circle the appropriate choice. Scratch paper is available if needed, though **NO** partial credit will be given for the Multiple Choice.

a. (4 pts) One of the following statements is NOT true:

(i) $\mathbf{P}(A \cup B) \leq \mathbf{P}(A) + \mathbf{P}(B)$
(ii) $\mathbf{P}(A \cup (B \backslash A)) = \mathbf{P}(A) + \mathbf{P}(B \backslash A)$
(iii) $\mathbf{P}(A \cap (B \backslash A))$ can be greater than 0
(iv) if $A \cap B$ is empty then $\mathbf{P}(A|B) \neq \mathbf{P}(A)$ (assume $\mathbf{P}(B) \neq 0$ and $\mathbf{P}(A) \neq 0$)

b. (4 pts) Two people each independently pick one integer between 1 and 5, inclusive. All choices are equally likely. Denote these numbers by X and Y. Then $\mathbf{P}(|X - Y| > 2 \mid X > 2)$ is equal to:

(i) 6/25
(ii) 1/5
(iii) 3/25
(iv) 2/5

c. (4 pts) There are n chairs placed in a perfect circle. The chairs are indistinguishable, and thus there is no identifiable start or stop to the circle. The number of ways n people can be placed in the chairs is

(i) $n!$
(ii) $(n-1)!$
(iii) n
(iv) $n-1$

d. (4 pts) When your friend Mike is asked if he smokes, he flips a fair coin. If the outcome is heads, he says no. If the outcome is tails, he tells the truth. Assume the probability that Mike smokes is p. Given Mike just said he doesn't smoke (he said no), the probability that the outcome of the coin flip is heads is

(i) p
(ii) $1/(1+p)$
(iii) $1/(1-p)$
(iv) $1/(2-p)$

e. (4 pts) Assume that X and Y are independent random variables. Let $Z = X^2 + Y^3$. Then

(i) $\text{var}(Z) = \text{var}(X^2) + \text{var}(Y^3)$
(ii) $\text{var}(Z) = (\text{var}(X))^2 + (\text{var}(Y))^3$
(iii) $\text{var}(Z) = \mathbf{E}[X^2] + \mathbf{E}[Y^3]$
(iv) $\text{var}(Z) = \text{var}(X^2) + \text{var}(Y^3) + 2\mathbf{E}[X^2]E(Y^3)$

f. (4 pts) Assume you roll a fair 6-sided die twice[1]. Define X to be the outcome of the first roll and Y the outcome of the second roll. Then

(i) $\mathbf{E}[XY] = (21/6)^2$
(ii) $\mathbf{E}[XY] = 0$
(iii) $\mathbf{E}[XY] = 91/36$
(iv) $\mathbf{E}[XY] = 91/6$

g. (4 pts) You roll a fair 6-sided die 100 times. Let X be the number of times a 6 appears. Then

(i) $\mathbf{E}[X^2] = 100/6 - 100/36$
(ii) $\mathbf{E}[X^2] = 100/6$
(iii) $\mathbf{E}[X^2] = 100/6 - 100/36 + (100/6)^2$
(iv) $\mathbf{E}[X^2] = 100/6 - (100/6)^2$

h. (4 pts) X refers to the time of the first arrival of a head in a coin flipping sequence (i.e. $X = 4$ means the first head was observed on the $4th$ flip). Assume the probability of a heads on any coin flip is p, and all flips are independent of each other. If Y is a random variable indicating the time of the second arrival of a head, then

(i) $\mathbf{E}[Y] = 1/p$
(ii) $\mathbf{E}[Y] = 1 + 1/p$
(iii) $\mathbf{E}[Y] = 1 + p$
(iv) $\mathbf{E}[Y] = 2/p$

i. (4 pts) If X is a binomial random variable where the probability of a success is p and the number of trials is denoted by n, then

(i) $\mathbf{P}(X \geq 2) = (1-p)^n - np^{n-1}$
(ii) $\mathbf{P}(X \geq 2) = 1 - p - np(1-p)$
(iii) $\mathbf{P}(X \geq 2) = 1 - (1-p)^n - np(1-p)^{n-1}$
(iv) $\mathbf{P}(X \geq 2) = p^n + np^{n-1}$

[1] A fair 6 sided die is labeled with the integers 1 through 6, with each side being equally likely to appear. Each roll of the die is independent of all other rolls.

j. (4 pts) You are in possession of n balls and s bins, where each bin is labeled 1 through s. You throw each ball into a bin at random, where each bin is equally likely and each throw is independent of any other throw. Let the random variable Y_i be defined as follows

$$Y_i = \begin{cases} 1, & \text{if bin } i \text{ is EMPTY} \\ 0, & \text{otherwise} \end{cases}$$

Let $Y = \sum_{i=1}^{s} Y_i$ be the total number of EMPTY bins. Then

(i) $\mathbf{E}[Y] = \dfrac{(s-1)^n}{s^{n-1}}$

(ii) $\mathbf{E}[Y] = \dfrac{1}{s^n}$

(iii) $\mathbf{E}[Y] = 1$

(iv) $\mathbf{E}[Y] = \dfrac{n(s-1)^n}{s^n}$

Question 2: Many of the parts in Question 2 can be answered independent of other parts.

Assume the following joint PMF for the discrete random variables X and Y.

y=3	1/6	1/9	0	1/9
y=2	0	0	2/9	0
y=1	1/6	0	1/9	1/9
	x=1	x=2	x=3	x=4

$p_{X,Y}(x, y)$, the joint PMF of X and Y.

a. (5 pts) Find the marginal PMF of X, $p_X(x)$.

Assume the following events:

$$\begin{aligned} A &: \{Y < 3\} \\ B &: \{X = 1\} \\ C &: \{X = 4\} \end{aligned}$$

b. (5 pts) Define the event D as $D = ((A \cap B^c) \cup C)^c$. Find $\mathbf{P}(D)$.

c. (6 pts) Find $\mathbf{E}[Y|A]$.

d. (6 pts) Define the event E as $E = (B \cup C)$. Conditioned on E are X and Y independent? Give a mathematical justification for your answer.

Question 3: Many parts of Question 3 can be answered independent of other parts. Part (g) is required for 6.431 students only.

As a member of the elite 6.041/6.431 student body you are entitled to shop at the local probabilistic grocery store. This is considered a great privilege. Not unexpectedly choosing fruit is far from deterministic. To obtain a piece of fruit you approach the fruit manager and ask him for a piece of fruit. He will give you an apple with probability 2/3 or an orange with probability 1/3. You always receive a piece of fruit after each request, and each request is independent of all other requests.

a. (5 pts) What is the probability you receive exactly 3 apples in your first 10 requests?

b. (6 pts) Let X be the number of requests you make until you receive your first orange, i.e. you receive your first orange on request number X. What is $\mathbf{E}[X]$?

c. (6 pts) Given you received 3 apples and 7 oranges in 10 requests, what is the probability that you received exactly 2 apples within the first 5 requests? *For full credit your final answer should contain no summations.*

You proceed to the check out to pay for your fruit. Prices are as follows: each apple is \$3, each orange is \$15. Let T be your total bill in dollars, assuming you have 10 pieces of fruit.

d. (6 pts) Find $\mathbf{E}[T]$

e. (6 pts) Find $\text{var}(T)$

You leave the grocery store and get on a bus with your friend Mais. The bus has k empty seats in a single row, where k is an integer greater than 2. You and Mais each choose a seat at random. Assume only one person can sit in a seat.

f. (6 pts) What is the probability you and Mais choose adjacent seats, i.e. you sit next to each other?

Part (g) is required for 6.431 students only. 6.041 students may attempt part (g), but will only receive extra credit if they have the correct answer. i.e. there is no partial credit for 6.041 students on part (g).

On any given day there are N shopping carts in the store, where N is a discrete uniform random variable from 1 to 100. All carts are labeled with sequential integers from 1 to N. Let event A be that immediately upon entering the store you see cart number 30. Also assume that the carts are randomly placed in the store such that you are equally likely to see any of the N carts when you enter the store.

g. (10 pts) Find the PMF of N conditioned on event A, $p_{N|A}(n)$.