Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2004)

Quiz 1 Solutions

1. The answer is the names of your TA and instructor.

2. (a) We can find c knowing that the probability of the entire sample space must equal 1.

$$\begin{aligned} 1 &= \sum_{x=1}^{3}\sum_{y=1}^{3} p_{X,Y}(x,y) \\ &= c+c+2c+2c+4c+3c+c+6c \\ &= 20c \end{aligned}$$

Therefore, $c = \frac{1}{20}$.

(b) $p_Y(2) = \sum_{x=1}^{3} p_{X,Y}(x,2) = 2c + 0 + 4c = 6c = \frac{3}{10}$

(c) $Z = YX^2$

$$\begin{aligned} \mathbf{E}[Z|Y=2] &= \mathbf{E}[YX^2|Y=2] \\ &= \mathbf{E}[2X^2|Y=2] \\ &= 2\mathbf{E}[X^2|Y=2] \end{aligned}$$

$p_{X|Y}(x|2) = \frac{p_{X,Y}(x,2)}{p_Y(2)}$
Therefore

$$p_{X|Y}(x|2) = \begin{cases} \frac{1/10}{3/10} = \frac{1}{3} & \text{if } x = 1 \\ \frac{1/5}{3/10} = \frac{2}{3} & \text{if } x = 3 \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \mathbf{E}[Z|Y=2] &= 2\sum_{x=1}^{3} x^2 p_{X|Y}(x|2) \\ &= 2((1^2)\cdot\frac{1}{3} + (3^2)\cdot\frac{2}{3}) \\ &= \frac{38}{3} \end{aligned}$$

(d) Yes. Given $X \neq 2$, the distribution of X is the same given Y=y.
$\mathbf{P}(X = x|Y = y, X \neq 2) = \mathbf{P}(X = x|X \neq 2)$
For example, $\mathbf{P}(X = 1|Y = 1, X \neq 2) = \mathbf{P}(X = 1|Y = 3, X \neq 2) = \mathbf{P}(X = 1|X \neq 2) = \frac{1}{3}$

(e) $p_{Y|X}(y|2) = \frac{p_{X,Y}(2,y)}{p_X(2)}$
$p_X(2) = \sum_{y=1}^{3} p_{X,Y}(2,y) = c + 0 + c = 2c = \frac{1}{10}$
Therefore

$$p_{Y|X}(y|2) = \begin{cases} \frac{1/20}{1/10} = \frac{1}{2} & \text{if } y = 1 \\ \frac{1/20}{1/10} = \frac{1}{2} & \text{if } y = 3 \\ 0 & \text{otherwise.} \end{cases}$$

$\mathbf{E}[Y^2|X=2] = \sum_{y=1}^{3} y^2 p_{Y|X}(y|2) = (1^2)\cdot\frac{1}{2} + (3^2)\cdot\frac{1}{2} = 5$
$\mathbf{E}[Y|X=2] = \sum_{y=1}^{3} y p_{Y|X}(y|2) = 1\cdot\frac{1}{2} + 3\cdot\frac{1}{2} = 2$
$\text{var}[Y|X=2] = \mathbf{E}[Y^2|X=2] - \mathbf{E}[Y|X=2]^2 = 5 - 2^2 = 1$

3. (a) Let H=toss was head, T=toss was tail

$$\begin{aligned}\mathbf{P}(B) &= \mathbf{P}(\{\text{1st 3 tosses were T, 4th was H, 7th was H, and last 3 tosses were T}\})\\ &= (1-p)^3\cdot p\cdot p\cdot(1-p)^3\\ &= (1-p)^6p^2\end{aligned}$$

(b) Yes.
$\mathbf{P}(F=4) = (1-p)^3p$
$\mathbf{P}(L=7) = p(1-p)^3$
$\mathbf{P}(L=7|F=4) = \frac{\mathbf{P}(F=4,L=7)}{P(F=4)} = \frac{\mathbf{P}(B)}{\mathbf{P}(F=4)} = \frac{(1-p)^6p^2}{(1-p)^3p} = p(1-p)^3 = \mathbf{P}(L=7)$

(c) F and L are not independent since if $F=f$, then $L\geq f$. For example, if you know $F=10$, you can conclude that $L=10$.

(d) Given that B occured, the possible sequences are T T T H _ _ H T T T where the fifth and sixth tosses can be either heads or tails. Let X be the total number of heads. Then
$\mathbf{P}(X=2|B) = \mathbf{P}(\{\text{T on 5th and 6th tosses}\}) = (1-p)^2$
$\mathbf{P}(X=3|B) = \mathbf{P}(\{\text{T on 5th toss, H on 6th toss}\}) + \mathbf{P}(\{\text{H on 5th toss, T on 6th toss}\}) = 2p(1-p)$
$\mathbf{P}(X=4|B) = \mathbf{P}(\{\text{H on 5th and 6th tosses}\}) = p^2$

$$\mathbf{P}(X=x|B) = \begin{cases}(1-p)^2 & \text{if } x=2\\ 2p(1-p) & \text{if } x=3\\ p^2 & \text{if } x=4\\ 0 & \text{otherwise}\end{cases}$$

(e) Let I_k be an indicator random variable at time k where

$$I_k = \begin{cases}1 & \text{if tosses resulted in H at times k-1 and k}\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\mathbf{P}_{I_k}(i) = \begin{cases}p^2 & \text{if } i=1\\ 1-p^2 & \text{if } i=0\end{cases}$$

So, $E(I_k) = p^2$. Note that $R = \sum_{k=2}^{10} I_k$, therefore

$$\begin{aligned}\mathbf{E}[R] &= \mathbf{E}[\sum_{k=2}^{10} I_k]\\ &= \sum_{k=2}^{10}\mathbf{E}[I_k] \text{ linearity of expectation}\\ &= 9\mathbf{E}[I_1] = 9p^2. \end{aligned} \quad (1)$$

(f) The sample space consists of all ways of getting 5 heads in 10 tosses, so it contains $\binom{10}{5}$ possible outcomes. We need to get 2 heads in the first 4 tosses and therefore 3 heads in the remaining 6 tosses to have a total of 5 heads in 10 tosses. Therefore, we have a total of $\binom{4}{2}\binom{6}{3}$ choices.
$\mathbf{P}(\{2 \text{ heads in first 4 tosses}\}|\{5 \text{ heads in 10 tosses}\}) = \frac{\binom{4}{2}\binom{6}{3}}{\binom{10}{5}}$

(g) Let N={number of heads in 10 tosses}. N is binomial with parameter p. Therefore
$p_N(n) = \binom{10}{n}p^n(1-p)^{10-n}, n = 0, 1, ..., 10$
$\mathbf{E}[N] = 10p$
$\text{var}[N] = 10p(1-p)$
$\mathbf{E}[N^2] = \text{var}[N] + \mathbf{E}[N]^2 = 10p(1-p) + 100p^2 = 10p + 90p^2$