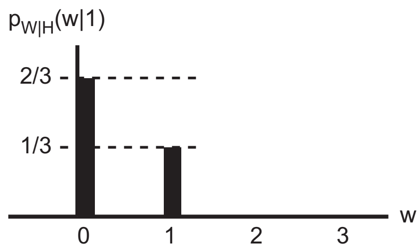
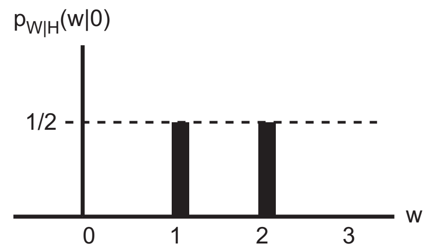
Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2005)

Quiz 1 Solutions

1. Random variable H takes on the value 1 with probability 1/3, and the value of 0 with probability 2/3. Random variable W is described by the conditional probabilities as follows:

$$p_{W|H}(w|1) = \begin{cases} 2/3, & w = 0 \\ 1/3, & w = 1 \\ 0, & \text{otherwise} \end{cases} \qquad p_{W|H}(w|0) = \begin{cases} 1/2, & w = 1 \\ 1/2, & w = 2 \\ 0, & \text{otherwise} \end{cases}$$





(a) Determine the joint PMF of H and W, i.e., $p_{H,W}(h, w)$.

$$p_{H,W}(h,w) = p_{W|H}(w|h) \cdot p_H(h) = \begin{cases} \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}, & h = 0, w = 1 \\ \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}, & h = 0, w = 2 \\ \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}, & h = 1, w = 0 \\ \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}, & h = 1, w = 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) Determine $p_{H|W}(0|1)$.

$p_{H|W}(0|1) = \frac{p_{H,W}(0,1)}{p_W(1)} = \frac{3}{4}$

(c) Are W and H independent? Explain why or why not.

No, W and H are not independent. The conditional PMFs of W conditioned on H have different support. i.e, $p_{W|H}(w|h)$ is non-zero when $w = 0,1$ if $h = 1$ and when $w = 1,2$ if $h = 0$.

It can also be quickly noted for example, that $p_{H|W}(0|1) = \frac{3}{4} \neq \frac{2}{3} = p_H(0)$.

2. Consider three events A, B and C. You know the following information about these events:

- $\mathbf{P}(A) = \mathbf{P}(B) = \mathbf{P}(C) = \frac{1}{10}$
- $\mathbf{P}(A \cap B \cap C) = \frac{1}{1000}$
- $\mathbf{P}(A \cap B^c) = \frac{9}{100}$
- $\mathbf{P}(B^c|C) = \frac{4}{5}$

(a) Are events A and B independent?

Yes, A and B are independent. We see this by:

$P(A \cap B) = P(A) - P(A \cap B^{\mathbf{C}}) = \frac{10}{100} - \frac{9}{100} = \frac{1}{100} = P(A)P(B)$

(b) Are events A, B and C independent?

No, A, B and C are not independent because at least one pairwise independence condition is violated, as seen below: $P(B^{\mathbf{c}} \cap C) = P(B^{\mathbf{c}}|C) \cdot P(C) = \frac{4}{5} \cdot \frac{1}{10} = \frac{4}{50}$

Now, $P(B \cap C) = P(C) - P(B^{\mathbf{c}} \cap C) = \frac{10}{100} - \frac{4}{50} = \frac{2}{100} \neq P(B) \cdot P(C)$.

(c) Compute $\mathbf{P}(A|B \cap C)$.

$P(A|B \cap C) = \frac{P(A \cap B \cap C)}{P(B \cap C)} = \frac{1/1000}{2/100} = \frac{1}{20}$.

3. You are playing a simple card game with another friend, using a standard 52 card deck: four suits (Diamonds, Hearts, Clubs and Spades), 13 cards of each suit (numbers from 2 to 10, Jack, Queen, King and Ace). An honest card dealer, different from you two, picks 8 cards out of 52 cards at random and divides them evenly between you two, 4 cards each.

(a) What is the probability that you get Ace, King, Queen and Jack of the Diamond suit?

There is only one combination of the $\binom{52}{4}$ combinations of cards you can get that give you Ace, King, Queen and Jack of the Diamond suit. Therefore, the probability is $\frac{1}{\binom{52}{4}}$.

(b) What is the probability that you get Ace, King, Queen and Jack of same suit?

One can get Ace, King, Queen and Jack of the same suit in 4 different ways because there are 4 suits. Therefore, the probability is $\frac{4}{\binom{52}{4}}$.

(c) What is the probability that you get Ace, King, Queen and Jack?

There are 4^4 different ways of getting Ace, King, Queen and Jack because there are 4 choices for each of Ace, King, Queen and Jack. Therefore, the probability is $\frac{4^4}{\binom{52}{4}}$.

(d) What is the probability that all four of your cards are of the Diamond suit?

Of the 13 Diamond cards in the deck, there are $\binom{13}{4}$ combinations of 4 of them. Therefore, the probability is $\frac{\binom{13}{4}}{\binom{52}{4}}$.

(e) You have Ace, King, Queen and Jack of Diamonds. What is the probability that all of the other player's cards are of the Diamond suit?

Since you have four of the Diamond cards, the other player can only possibly have 4 of the remaining 9. This implies there are $\binom{9}{4}$ ways in which the other player can have all Diamonds. The total possible number of combinations of cards he could have is $\binom{48}{4}$ because you already have the Ace, King, Queen and Jack of Diamonds. Therefore, the probability is $\frac{\binom{9}{4}}{\binom{48}{4}}$.

4. Oscar is a used car salesman who is always trying to maximize his earnings, but often loses customers by trying to rush their decision. His manager decides to help him spend more time with each customer by limiting the number of customers Oscar is allowed to interact with during the day. He presents Oscar with two different options:

Strategy 1: Have one distinct customer interaction per 15 minute slot, where there is a total of 32 slots per day. The probability of a sale for each interaction is 1/4, and all customer interactions and sales are independent of each other. Each sale yields a commission $c_1 = \$80$.

Strategy 2: Have one distinct customer interaction per 30 minute slot, where there is a total of 16 slots per day. The probability of a sale for each interaction is 1/2, and all customer interactions and sales are independent of each other. Each sale yields a commission $c_2 = \$100$.

Let K be the number of cars Oscar sells on a particular day using Strategy 1. His earnings on that day are $Q_1 = K \cdot c_1$. Let M be the number of cars Oscar sells on a particular day using Strategy 2. His earnings on that day are $Q_2 = M \cdot c_2$.

Question (f) can be solved independently of (d) and (e). Also, question (g) can be solved independently of (d), (e) and (f).

(a) Determine the PMF of K.

The PMF is binomial with $p = 1/4$ and $n = 32$:

$$p_K(k) = \begin{cases} \binom{32}{k}(\frac{1}{4})^k(\frac{3}{4})^{32-k}, & \text{for } k = 0, 1, ..., 32 \\ 0, & \text{otherwise} \end{cases}$$

(b) Find the expectation and the variance of K. Provide numerical answers.

$E[K] = np = 32 \cdot \frac{1}{4} = 8$ cars

$Var(K) = np(1-p) = 32 \cdot \frac{1}{4} \cdot \frac{3}{4} = 6$ cars^2

(c) Determine the expected earnings for each strategy and state which choice Oscar should take to maximize his expected earnings.

First we find $E[M] = 16 \cdot \frac{1}{2} = 8$ cars
$E[Q_1] = E[K \cdot c_1] = c_1 \cdot E[K] = \$640$
$E[Q_2] = E[M \cdot c_2] = c_2 \cdot E[M] = \$800$

Therefore, Strategy 2 optimizes Oscar's earnings.

(d) Despite the above analysis, Oscar can't make up his mind of which choice to take. He decides to flip a fair coin and follow Strategy 1 if it turns up heads, or follow Strategy 2 if it turns up tails. Let L be the number of cars he sells using this strategy. Determine the PMF and the expectation of L. Provide a numerical answer for the expectation.

$$\begin{aligned} p_L(l) &= P(\text{Oscar sells } L \text{ cars}) \\ &= P(\text{Oscar sells } L \text{ cars}|\text{Heads}) \cdot P(\text{Heads}) + P(\text{Oscar sells } L \text{ cars}|\text{Tails}) \cdot P(\text{Tails}) \\ &= p_K(l) \cdot \frac{1}{2} + p_M(l) \cdot \frac{1}{2}. \end{aligned}$$

Now,

$$p_L(l) = \begin{cases} \frac{1}{2}\binom{32}{l}(\frac{1}{4})^l(\frac{3}{4})^{32-l} + \frac{1}{2}\binom{16}{l}(\frac{1}{2})^l(\frac{1}{2})^{16-l}, & l = 0, 1, ..., 16 \\ \frac{1}{2}\binom{32}{l}(\frac{1}{4})^l(\frac{3}{4})^{32-l}, & l = 17, 18, ..., 32 \\ 0, & \text{otherwise} \end{cases}$$

By the total expectation theorem,

$E[L] = E[L|\text{Heads}] \cdot P(\text{Heads}) + E[L|\text{Tails}] \cdot P(\text{Tails}) = E[K] \cdot \frac{1}{2} + E[M] \cdot \frac{1}{2} = 8$ cars.

(e) Oscar is still using his randomized strategy from part d. Given that he sold 4 cars one day, what is the probability that the coin turned up heads? If he had instead sold 18 cars, what would be the probability that the coin turned up heads?

$$\begin{aligned} P(\text{Heads}|L=4) &= \frac{P(L=4 \text{ and Heads})}{P(L=4)} = \frac{P(L=4|\text{Heads}) \cdot P(\text{Heads})}{P(L=4)} \\ &= \frac{\frac{1}{2}\binom{32}{4}(\frac{1}{4})^4(\frac{3}{4})^{28}}{\frac{1}{2}\binom{32}{4}(\frac{1}{4})^4(\frac{3}{4})^{28} + \frac{1}{2}\binom{16}{4}(\frac{1}{2})^{16}} \end{aligned}$$

Now, for Oscar to sell 18 cars in one day he could have only flipped heads. Therefore,

$P(\text{Heads}|L = 18) = 1$

(f) This question can be solved independently of (d) and (e). Suppose that on one particular day, Oscar decides to follow strategy one. On this same day, Oscar's manager decides to raise the commission for strategy one to either \$90 or \$110, and since he is as indecisive as Oscar, he flips a fair coin once to decide which raise to use for the entire day. Let C be the random variable for the new commission. Find the expectation and the variance of Oscar's earnings $Q = K \cdot C$ that day. Provide numerical answers.

First, we compute the expectation and the second moment of C:
$E[C] = 90 \cdot \frac{1}{2} + 110 \cdot \frac{1}{2} = 100$ dollars,
$E[C^2] = 8100 \cdot \frac{1}{2} + 12100 \cdot \frac{1}{2} = 10100$ dollars^2.
Since K and C are independent,

$E[Q] = E[K \cdot C] = E[K] \cdot E[C] = 800$ dollars.
$E[Q^2] = E[K^2 \cdot C^2] = E[K^2] \cdot E[C^2] = (Var(K) + (E[K])^2) \cdot E[C^2]$
$= (6 + 64) \cdot 10100 = 70 \cdot 10100 = 707000$ dollars^2.
$Var(Q) = E[Q^2] - (E[Q])^2 = 707000 - 640000 = 67000$ dollars^2.

(g) This question can be solved independently of (d), (e) and (f). Suppose Oscar now decides to always follow strategy one. Let X be the number of unsuccessful customer interactions (i.e. interactions which did not result in a sale) BEFORE Oscar makes his first sale. Find the PMF, and numerical values for the expectation, and the variance of X (Careful, there is zero probability that a standard geometric random variable takes on the value of zero, but X will be zero if Oscar sells to his first customer).

Let Y be the number of customer interactions up to and including the first sale. Y is a geometric RV with $p = 1/4$: $p_Y(y) = \begin{cases} (\frac{3}{4})^{y-1}(\frac{1}{4}), & y = 1, 2, 3, ... \\ 0, & \text{otherwise} \end{cases}$

$E[Y] = \frac{1}{p} = 4$, $Var(Y) = \frac{1-p}{p^2} = 12$.

Noting that $X = Y - 1$, we obtain

$$p_X(x) = P(X = x) = P(Y = x + 1) = \begin{cases} (\frac{3}{4})^x(\frac{1}{4}), & x = 0, 1, 2, ... \\ 0, & \text{otherwise} \end{cases}$$

$E[X] = E[Y] - 1 = 3$, $Var(X) = Var(Y) = 12$.