6.431 Quiz 1 Solutions
Handed out: October 10, 2001

Problem 1. (50 points)

(a) The races are independent

(i) Since the races are independent and we're interested only in the event of winning a game or loosing it, we can think of the problem as that of computing the probability of k successes among n tries, where each try is an independent identically distributed Bernoulli like random variable.

Let's start by the computing the probability of success (making money) in one game. Alok will make money in a game, if and only if horses 1, 2 and 3 finish in the first three positions (irrespective of the order). The probability p the ordered finish is (1,2,3), by the definition of p_1, p_2 and p_3 is given by: $p = p_1.p_2.p_3$. Note next that the probability of any ordered finish (for any trio of horses for that matter) is the same! Hence we 3! possible favorable finishes, consequenty, the probability of Alok making money on one game is $(3!)p = 6p_1.p_2.p_3 = 6/(6.5.4) = 1/20$.

As it has been argued before, the number of successes is Binomial and hence

$$\begin{aligned}\mathbf{P}(\text{making money on } k \text{ races among } n) &= \binom{n}{k}(6p)^k(1-6p)^{(n-k)} \\ &= \binom{n}{k}\left(\frac{1}{20}\right)^k\left(\frac{19}{20}\right)^{(n-k)}\end{aligned}$$

(ii) Consider first Alok's wealth after a game. It either increases after a win or he looses a dollar after a loss. Denote by X_i the random variable representing Alok's gains/losses in one game. If he looses $X_i = -1$. On the other hand, given that he won a game, with probability 1/6, $X_i = 14$ (if the winning horses finish in the same order he gambled on, after betting one dollar he will receive 15, which means that his wealth increases by 14 dollars). Similarly, given that he won, with probability 5/6 it is equal to 2.

His total gains during n races are equal to $\sum_{i=1}^{n} X_i$. Using the linearity of expectation, given that he won k races, his expected gain is $\sum_{i=1}^{k} \mathsf{E}[X_i| \text{ he won }] - (n-k) = \sum_{i=1}^{k}(14/6 + 10/6) - (n-k) = 5k - n$.

(b) In this section the races are dependent

(i) Since Alok's strategy is different for the first game from the rest, we will consider two cases. For the first game, his probability of winning is equal to the value $6p$ we have computed (a,i) ($p = 1/120$).

For the remaining games, Alok will win if the winning horses of the previous game, say horses l, j and i, finish in the top three. Note that, as before, the finishing orders of horses l, j and i are equally likely. In particular, the probability of a (l, j, i) finish is equal to that of a (j, l, i) finish and so on.

Given the result (l, j, k) of the previous race, these horses will finish in the top three in a particular order with probability q. There are 3! such possibilities among $6!/3! = 120$ choices. Hence, $6q + 114q/6 = 1$ which implies that $q = 6/150 = 1/25$.

Hence, the probability that Alok makes money on any race (apart from the first) is equal to $6q = 6/25$.

In summary, Alok will win exactly k races among n if

- He wins the first race and then wins $k-1$ races among the remaining $n-1$
- Or he looses the first race and then wins k among the remaing n.

$$\begin{aligned}
\mathbf{P}(\text{winning } k \text{ among } n) &= (6p)\binom{n-1}{k-1}(6q)^{k-1}(1-6q)^{(n-k)} + \\
&\quad (1-6p)\binom{n-1}{k}(6q)^{k}(1-6q)^{(n-1-k)} \\
&= \frac{1}{20}\binom{n-1}{k-1}\left(\frac{6}{25}\right)^{k-1}\left(\frac{19}{25}\right)^{(n-k)} + \\
&\quad \frac{19}{20}\binom{n-1}{k}\left(\frac{6}{25}\right)^{k}\left(\frac{19}{25}\right)^{(n-1-k)}
\end{aligned}$$

(ii) The answer to this question can be readily obtained using Bayes' rule. Let A be the event that the winning horses of the first race are $(1,2,3)$ (in this order). Define B to be the event the winning horses of the second race are $(1,2,3)$ (in this order). We need to find $\mathbf{P}(A|B)$.

$$\begin{aligned}
\mathbf{P}(A|B) &= \frac{\mathbf{P}(B|A)\mathbf{P}(A)}{\mathbf{P}(B)} \\
&= \frac{qp}{\mathbf{P}(B)}
\end{aligned}$$

It remains to determine $\mathbf{P}(B)$. For this purpose we use the total probability theorem. Let E be the event that $[1,2,3]$ won the first race (in any order).

$$\begin{aligned}
\mathbf{P}(B) &= \mathbf{P}(B \cap E) + \mathbf{P}(B \cap E^c) \\
&= \mathbf{P}(B|E)\mathbf{P}(E) + \mathbf{P}(B|E^c)\mathbf{P}(E^c) \\
&= q(6p) + (q/6)(1-6p) \\
&= \frac{1}{25.20} + \frac{19}{150.20} \\
&= \frac{1}{120}
\end{aligned}$$

which is in accordance with one's intuition that, since all the outcomes are equally likely for the first race, there is no reason to believe that they won't be for the second race.

Hence, $\mathbf{P}(A|B) = 1/25$.

Problem 2. (50 points)

(a) In order to determine the PMF of Z, we compute the probability of the event that $Z = k$:

$$\begin{aligned}
p_Z(k) &= \mathbf{P}(Z = k) && (1)\\
&= \sum_l \mathbf{P}(Z = k \text{ and } X_1 = l) && (2)\\
&= \sum_{l\le k} \mathbf{P}(Z = k \text{ and } X_1 = l) && (3)\\
&= \sum_{l\le k} \mathbf{P}(Z = k | X_1 = l)\mathbf{P}(X_1 = l) && (4)\\
&= \sum_{l\le k} \mathbf{P}(X_2 = k - l | X_1 = l)\mathbf{P}(X_1 = l) && (5)\\
&= \sum_{l\le k} \mathbf{P}(X_2 = k - l)\mathbf{P}(X_1 = l) && (6)\\
&= \sum_{l\le k} p_{X_2}(k - l)p_{X_1}(l) && (7)\\
&= \sum_{l\le k} \frac{\mu^{k-l}e^{-\mu}}{(k-l)!}\frac{\mu^l e^{-\mu}}{l!} && (8)\\
&= \mu^k e^{-2\mu} \sum_{l\le k} \frac{1}{(k-l)!l!} && (9)\\
&= \mu^k e^{-2\mu} \frac{1}{k!} \sum_{l\le k} \frac{k!}{(k-l)!l!} && (10)\\
&= \mu^k e^{-2\mu} \frac{1}{k!} \sum_{l\le k} \binom{k}{l} && (11)\\
&= \mu^k e^{-2\mu} \frac{1}{k!} 2^k && (12)\\
&= \frac{(2\mu)^k e^{-2\mu}}{k!}, && (13)
\end{aligned}$$

where equation 2 is a direct application of the total probability theorem. Equation 3 is readily obtained by realizing that the event $Z = k$ and $X_1 = l$ where $l > k$ has probability zero. Equation 4 is a direct application of the definition of conditional probabilities, and equation 6 is due to the fact that X_2 and X_1 are independent.

(b) We can prove by induction on n that the PMF of $Z = X_1 + X_2 + \ldots + X_n$ is

$$p_Z(k) = \frac{(n\mu)^k e^{-(n\mu)}}{k!}, \quad k = 0, 1, 2, \ldots$$

The hypothesis is correct for $n = 2$. Assume that it is correct upto $n - 1$ and let's prove that it is correct for n.

Define $Y = X_1 + X_2 + \ldots + X_{n-1}$, therefore $Z = Y + X_n$. Furthermore, since the X_i's are independent, X_n is independent of any function of $X_1, \ldots, X_{n-1}$ and in particular Y. In summary, Z is the sum of two independent random variables Y and X_n and following the same steps as before

$$\begin{aligned}
p_Z(k) &= \mathbf{P}(Z = k) \\
&= \sum_{l \leq k} \mathbf{P}(Z = k \text{ and } X_n = l) \\
&= \sum_{l \leq k} \mathbf{P}(Z = k | X_n = l)\mathbf{P}(X_n = l) \\
&= \sum_{l \leq k} \mathbf{P}(Y = k - l | X_n = l)\mathbf{P}(X_n = l) \\
&= \sum_{l \leq k} \mathbf{P}(Y = k - l)\mathbf{P}(X_n = l) \\
&= \sum_{l \leq k} p_Y(k - l) p_{X_n}(l)
\end{aligned}$$

Since the PMF of Y satisfies the hypothesis,

$$\begin{aligned}
p_Z(k) &= \sum_{l \leq k} \frac{((n-1)\mu)^{k-l} e^{-(n-1)\mu}}{(k-l)!} \frac{\mu^l e^{-\mu}}{l!} \\
&= \mu^k e^{-n\mu} \sum_{l \leq k} \frac{1}{(k-l)! l!} (n-1)^{k-l} \\
&= \mu^k e^{-n\mu} \frac{1}{k!} \sum_{l \leq k} \frac{k!}{(k-l)! l!} (n-1)^{k-l} \\
&= \mu^k e^{-n\mu} \frac{1}{k!} \sum_{l \leq k} \binom{k}{l} (n-1)^{k-l} \\
&= \mu^k e^{-n\mu} \frac{1}{k!} n^k \\
&= \frac{(n\mu)^k e^{-n\mu}}{k!}
\end{aligned}$$

(c) $T = X_1 + X_2 + \cdots + X_N$. Note that the number of variables added is a random variable (N). Using the total probability theorem and conditional probabilities

$$\begin{aligned}
p_T(k) &= \sum_n \mathbf{P}(T = k \text{ and } N = n) \\
&= \sum_n \mathbf{P}(T = k | N = n)\mathbf{P}(N = n)
\end{aligned}$$

The expression of the first term is given in (b), while the PMF of N is defined in the problem statement. Hence

$$\begin{aligned}
p_T(k) &= \sum_{n \geq 1} \frac{(n\mu)^k e^{-n\mu}}{k!} \frac{\lambda^n e^{-\lambda}}{n!} \\
&= \frac{\mu^k e^{-\lambda}}{k!} \sum_{n \geq 1} \frac{n^k e^{-n\mu} \lambda^n}{n!}
\end{aligned}$$

(d) $T = X_1 + X_2 + \cdots + X_N$. Using the total expectation theorem,

$$\begin{aligned}
\mathsf{E}[T] &= \mathsf{E}[\mathsf{E}[T|N]] \\
&= \mathsf{E}\left[\sum_{i=1}^{N} \mathsf{E}[X_i]\right] \\
&= \mathsf{E}\left[\sum_{i=1}^{N} \mu\right] \\
&= \mathsf{E}[N\mu] \\
&= \mu \mathsf{E}[N] \\
&= \mu\lambda,
\end{aligned}$$

where we have used the fact that expectation is linear, and that the expected value of a Poisson random variable is equal to its parameter.

(e) In order to compute the variance, we compute $\mathsf{Var}[T] = \mathsf{E}[T^2] - \mathsf{E}[T]^2$. We only need to determine the first term. Using the same approach as in (d),

$$\begin{aligned}
\mathsf{E}\left[T^2\right] &= \mathsf{E}\left[\mathsf{E}\left[T^2|N\right]\right] \\
&= \mathsf{E}\left[\mathsf{Var}[T|N] + \mathsf{E}[T|N]^2\right]
\end{aligned}$$

Now

$$\begin{aligned}
\mathsf{E}[\mathsf{Var}[T|N]] &= \mathsf{E}\left[\sum_{i=1}^{N} \mathsf{Var}[X_i]\right] \\
&= \mathsf{E}\left[\sum_{i=1}^{N} \mu\right] \\
&= \mathsf{E}[N\mu] \\
&= \mu \mathsf{E}[N] \\
&= \mu\lambda,
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{E}\left[\mathsf{E}[T|N]^2\right] &= \mathsf{E}\left[(N\mu)^2\right] \\
&= \mu^2(\mathsf{Var}[N] + \mathsf{E}[N]^2) \\
&= \mu^2(\lambda + \lambda^2)
\end{aligned}$$

In summary

$$\begin{aligned}
\mathsf{Var}[T] &= \mathsf{E}\left[T^2\right] - \mathsf{E}[T]^2 \\
&= \mu\lambda + \mu^2(\lambda + \lambda^2) - \mu^2\lambda^2 \\
&= \mu\lambda(1 + \mu)
\end{aligned}$$