MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2004)

6.041 Fall 2004 Quiz 2 Solutions

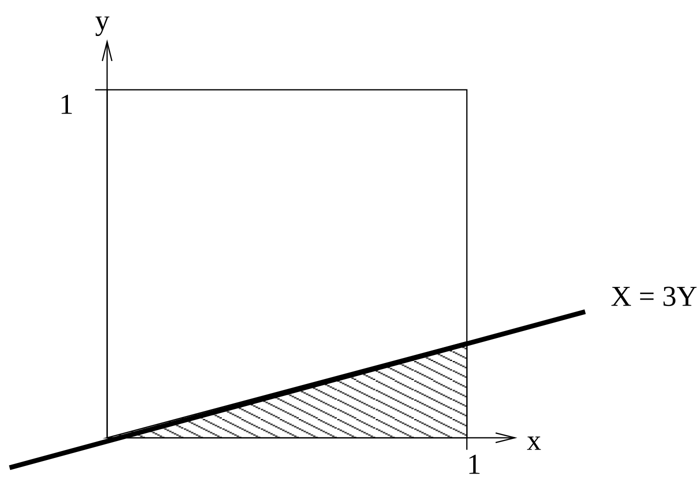
Tuesday 9th of November, 7:30-9:30pm

Problem 2. (55 points)

Let X and Y be independent uniform random variables with range $[0, 1]$. In particular, the variance of X and Y is $1/12$.

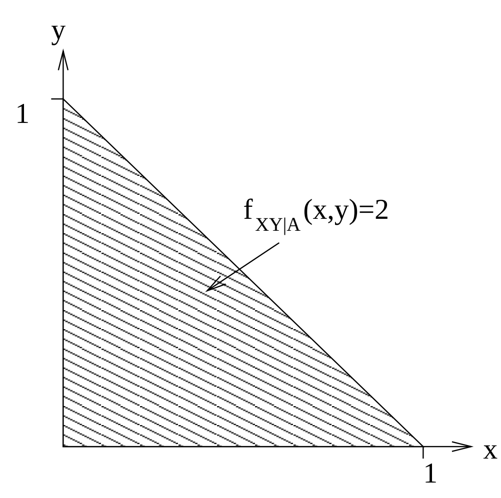
(7 pts) **2(a)** Find the mean and variance of $X - 3Y$.
$\mathbf{E}[X - 3Y] = \mathbf{E}[X] - 3\mathbf{E}[Y] = 1/2 - 3/2 = -1$
$\text{var}(X - 3Y) = \text{var}(X) + 9\text{var}(Y) = 1/12 + 9/12 = 10/12 = 5/6$

(8 pts) **2(b)** Find the probability that $X \geq 3Y$.
$\mathbf{P}[X \geq 3Y] = 1/2 \cdot 1 \cdot 1/3 \cdot 1 = 1/6$



(7 pts) **2(c)** Find the conditional joint PDF of X and Y, given that the event $X + Y \leq 1$ has occurred.
Event A = $X + Y \leq 1$.

$$f_{X,Y|A}(x, y) = \frac{f_{X,Y}(x, y)}{P(A)} = \frac{f_{X,Y}(x, y)}{1/2} = \begin{cases} 2, & x \in [0, 1], y \in [0, 1 - x] \\ 0, & \text{otherwise} \end{cases}$$



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(10 pts) **2(d)** Find the conditional PDF of $(X+Y)^2$, given that the event $X+Y \leq 1$ has occurred.
Let's define $Z = (X+Y)^2$, and Event A $= X+Y \leq 1$.

$$P(Z \leq z|A) = P((X+Y)^2 \leq z|A) = P(-\sqrt{z} \leq X+Y \leq \sqrt{z}|A) = \begin{cases} 2 \cdot (1/2) \cdot \sqrt{z}^2, & z \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

$$F_{Z|A}(z) = \begin{cases} z, & z \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

$$f_{Z|A}(z) = \begin{cases} 1, & z \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

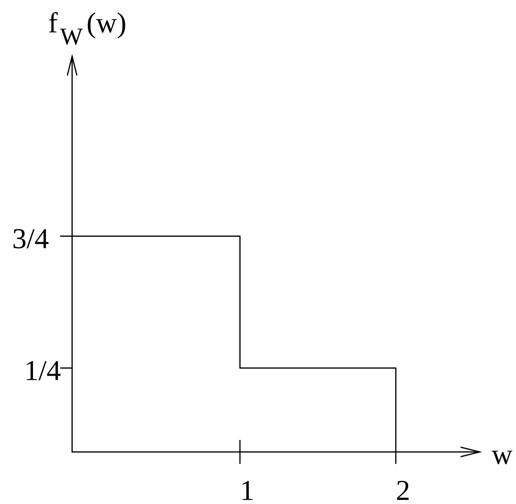
(7 pts) **2(e)** Provide a fully labelled sketch of the conditional PDF of XY given that $Y = 0.5$.
Let Z=XY.

$$f_{Z|Y}(z|.5) = \begin{cases} 2, & 0 \leq z \leq 1/2 \\ 0, & \text{otherwise} \end{cases}$$

(16 pts) **2(f)** A random variable W is defined as follows. We toss a fair coin (independent of X and Y). If the result is "heads", we let $W = X$; if it is tails, we let $W = 2Y$.

(8 pts) **(i)** Provide a fully labelled sketch of the PDF of W.

$$f_W(w) = 1/2 \cdot f_X(w) + 1/2 \cdot f_{2Y}(w) = 1/2 \cdot f_X(w) + 1/4 \cdot f_Y(\frac{w}{2})$$



(8 pts) **(ii)** Find the probability of "heads" given that $0.5 \leq W \leq 1.5$.

$$P[\text{heads}|0.5 \leq W \leq 1.5] = \frac{P(0.5 \leq W \leq 1.5|\text{heads})\mathbf{P}(heads)}{\mathbf{P}(0.5 \leq W \leq 1.5|\text{heads})\mathbf{P}(heads) + \mathbf{P}(0.5 \leq W \leq 1.5|\text{tails})\mathbf{P}(tails)}$$

$$= \frac{\mathbf{P}(0.5 \leq X \leq 1.5)\frac{1}{2}}{P(0.5 \leq X \leq 1.5)\frac{1}{2} + \mathbf{P}(0.5 \leq 2Y \leq 1.5)\frac{1}{2}} = \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}} = 1/2$$

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Problem 3. (43 points) Let $X_1, X_2, \ldots$ be independent normal random variables with mean 2 and variance 4. Let N be a geometric random variable which is independent of the X_i, with parameter $p = 2/3$. (In particular, $\mathbf{E}[N] = 3/2$ and $\mathbf{E}[N^2] = 3$.)

(8 pts) **3(a)** If δ is a small positive number, we have $\mathbf{P}(|X_1| \leq \delta) \approx \alpha\delta$, for some constant α. Find the value of α. (Your answer may involve π, no need to evaluate numerically.)

$\mathbf{P}(|X_1| \leq \delta) \approx \alpha\delta$

$\mathbf{P}(-\delta \leq X_1 \leq \delta) = \int_{-\delta}^{\delta} f_{X_1}(x)dx_1 = 2\delta \cdot f_{X_1}(0) = \delta \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}}$

$\alpha = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}}$

(8 pts) **3(b)** Find $\mathbf{E}[X_1 N]$.

$\mathbf{E}[X_1 N] = \mathbf{E}[X_1]\mathbf{E}[N] = \frac{3}{2} \cdot 2 = 3$

(8 pts) **3(c)** Find the variance of $X_1 N$.

$\mathrm{var}(X_1 N) = \mathbf{E}(X_1^2 N^2) - \mathbf{E}(X_1 N)^2 = (4+4)3 - 3^2 = 15$

(8 pts) **3(d)** Find $\mathbf{E}[X_1 + \cdots + X_N \mid N \geq 2]$.

$$\begin{aligned}
\mathbf{E}[X_1 + \cdots + X_N] &= \mathbf{E}[X_1 + \cdots + X_N \mid N \geq 2]\mathbf{P}(N \geq 2) + \mathbf{E}[X_1 + \cdots + X_N \mid N < 2]\mathbf{P}(N < 2) \\
3 &= \mathbf{E}[X_1 + \cdots + X_N \mid N \geq 2](1-p) + \mathbf{E}[X_1](p) \\
\mathbf{E}[X_1 + \cdots + X_N \mid N \geq 2] &= 3(3 - 2(2/3)) = 5
\end{aligned}$$

(8 pts) **3(e)** Write down the transform associated with $N + X_1 + \cdots + X_N$.

Let $Z = N + X_1 + \cdots + X_N$. Note that N and $X_1 + \cdots + X_N$ are NOT independent.

$M_Z(s) = E[E[e^{s(N+X_1+\cdots+X_N)}|N]] = E[E[e^{sN} \cdot e^{s(X_1+\cdots+X_N)}|N]] = E[e^{sN}(M_X(s))^N]$
$= E[(e^s M_X(s))^N] = M_N(s)|_{e^s = e^s M_X(s)}$

$M_N(s) = \frac{(2/3)e^s}{1-(1/3)e^s}$

$M_X(s) = e^{2s^2+2s}$

$M_Z(s) = \frac{(2/3)e^s e^{2s^2+2s}}{1-(1/3)e^s e^{2s^2+2s}} = \frac{2e^{2s^2+3s}}{3-e^{2s^2+3s}}$

Alternative Solutions to Fall 2004, Quiz 2, Problems 3(d) and 3(e)

Vivek Goyal

3(d) We wish to compute

$$E[X_1 + X_2 + \cdots + X_N \mid N \geq 2].$$

The primary solution uses the law of total expectation to expand $E[X_1 + X_2 + \cdots + X_N \mid N \geq 2]$ in terms of the desired quantity and several easily-computed quantities. An alternative solution explicitly uses the conditional PMF of N with conditioning on the event $N \geq 2$.

Denote by K the (random) number of terms in the sum. Because of the memoryless property of the geometric PMF, $K - 1$ is a geometric random variable with parameter 2/3. Recall that this means in particular that $E[K-1] = \frac{3}{2}$. We have

$$E[X_1 + X_2 + \cdots + X_N \mid N \geq 2] = E[X_1 + X_2 + \cdots + X_K] = E[K] \cdot E[X_i] = \left(1 + \frac{3}{2}\right) 2 = 5.$$

3(e) This problem does *not* allow direct use of the theory established to analyze sums of independent random variables because N and $X_1 + X_2 + \cdots + X_N$ are not independent. The primary solution circumvents this by using iterated expectation, with conditioning on N. An alternative is to define new random variables so that formulas from the book can be applied directly.

For each i, let $Y_i = 1 + X_i$. Then the Y_is are independent and

$$Y_1 + Y_2 + \cdots + Y_N = (1 + X_1) + (1 + X_2) + \cdots + (1 + X_N) = N + X_1 + X_2 + \cdots + X_N = Z.$$

This allows us to now use the theory established to analyze the sum of N independent, identically distributed random variables, where N is a random variable that is independent of the terms in the sum.

$$M_{Y_i}(s) = E[e^{sY_i}] = E[e^{s(X_i+1)}] = e^s E[e^{sX_i}] = e^s e^{2s^2+2s} = e^{2s^2+3s}$$

$$M_N(s) = \frac{\frac{2}{3}e^s}{1 - \frac{1}{3}e^s}$$

$$M_Z(s) = M_N(s)|_{e^s = M_Y(s)} = \frac{\frac{2}{3}e^{2s^2+3s}}{1 - \frac{1}{3}e^{2s^2+3s}}$$