Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2008)

Recitation 3: Answers
September 11, 2008

1. (a) Intuitively, the events A and B are not independent because each depends on the second coin toss. For a convincing formal answer, make the following computations, where H_i is the event that the ith toss is heads:

$$\begin{aligned}
\mathbf{P}(A) &= \mathbf{P}(H_1, H_2) \stackrel{(a)}{=} \mathbf{P}(H_1)\mathbf{P}(H_2) = p \cdot p = p^2 \\
\mathbf{P}(B) &= \mathbf{P}(H_2, H_3) \stackrel{(b)}{=} \mathbf{P}(H_2)\mathbf{P}(H_3) = p \cdot p = p^2 \\
\mathbf{P}(A \cap B) &= \mathbf{P}(H_1, H_2, H_3) \stackrel{(c)}{=} \mathbf{P}(H_1)\mathbf{P}(H_2)\mathbf{P}(H_3) = p \cdot p \cdot p = p^3
\end{aligned}$$

where (a), (b), and (c) follow from the independence of the coin tosses. This seems to show that $\mathbf{P}(A \cap B) \neq \mathbf{P}(A)\mathbf{P}(B)$.

We should actually be slightly more careful. If $p = 0$ or $p = 1$, then the events A and B are independent; but this is a technicality—in the reasonable cases where $p \in (0, 1)$, the events are not independent.

(b) It is clear that events A and C are independent since they depend on disjoint subsets of the coin tosses and the coin tosses are independent. Computations similar to those above can be used to provide a more formal answer.

2. (a) Hare must score 3 points in each round to win a two-round game. Using the independence of the rounds of the game, the probability that Hare wins a two-round game is p^2.

(b) We are asked to find p such that $p^2 = 1 - p$. The solution to this quadratic equation is the inverse of the golden ratio: $p = (\sqrt{5} - 1)/2 \approx 0.618$. With this value of p, Hare wins a one-round game with probability ≈ 0.618 but wins a two-round game with probability only ≈ 0.382.

The cited paper by Cover provides an example in which Hare's probability of winning a one-round game is arbitrarily close to 1 but for a long enough game is arbitrarily close to 0. There are also examples in which the probability of winning depends strongly on whether the game has an odd or even number of rounds.

3. Problem 1.31, page 60 of text. See online solutions.