Recitation 4: Answers
September 16, 2008

1. The sample space consists of all possible choices for the birthday of each person. Since there are n persons, and each has 365 choices for their birthday, the sample space has 365^n elements. Let us now consider those choices of birthdays for which no two persons have the same birthday. Assuming that $n \leq 365$, there are 365 choices for the first person, 364 for the second, etc., for a total of $365 \cdot 364 \cdots (365 - n + 1)$. Thus,

$$\mathbf{P}(\text{no two birthdays coincide}) = \frac{365 \cdot 364 \cdots (365 - n + 1)}{365^n}.$$

It is interesting to note that for n as small as 23, the probability that there are two persons with the same birthday is larger than 1/2.

2. (a) There are n choices for the club leader. Once the leader is chosen, we are left with a set of $n - 1$ available persons, and we are free to choose any of the 2^{n-1} subsets.

   (b) We can form a k-person club by first selecting k out of the n available persons [there are $\binom{n}{k}$ choices], and then selecting one of the members to be the leader (there are k choices). Thus, there is a total of $k\binom{n}{k}$ k-person clubs. We then sum over all k to obtain the number of possible clubs of any size.

3. Problem 1.61, page 69 of the text. See solution in text.

4. (a) The N_is are the numbers of times each ball is selected, so the sum of the N_is must be the total number of draws from the urn.

   (b) There is a nice visualization for this. Make a dot for each drawn ball, grouped according the ball's identity:

$$\underbrace{\cdots}_{N_1}\ \underbrace{\cdots}_{N_2}\ \underbrace{\quad}_{\cdots}\ \underbrace{\cdots}_{N_n}$$

   There is a total of k dots put in n groups. Think of there being a separator mark between groups, so there are $n - 1$ separator marks:

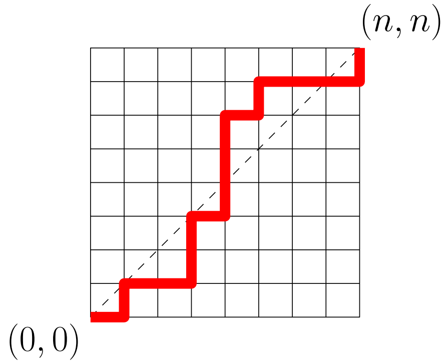
$$\underbrace{\cdots}_{N_1}\ |\ \underbrace{\cdots}_{N_2}\ |\ \underbrace{\quad}_{\cdots}\ |\ \underbrace{\cdots}_{N_n}$$

   This gives a grand total of $k + n - 1$ dots and marks. The number of solutions is the number of ways to place k dots in $k + n - 1$ locations: $\binom{k+n-1}{k}$.

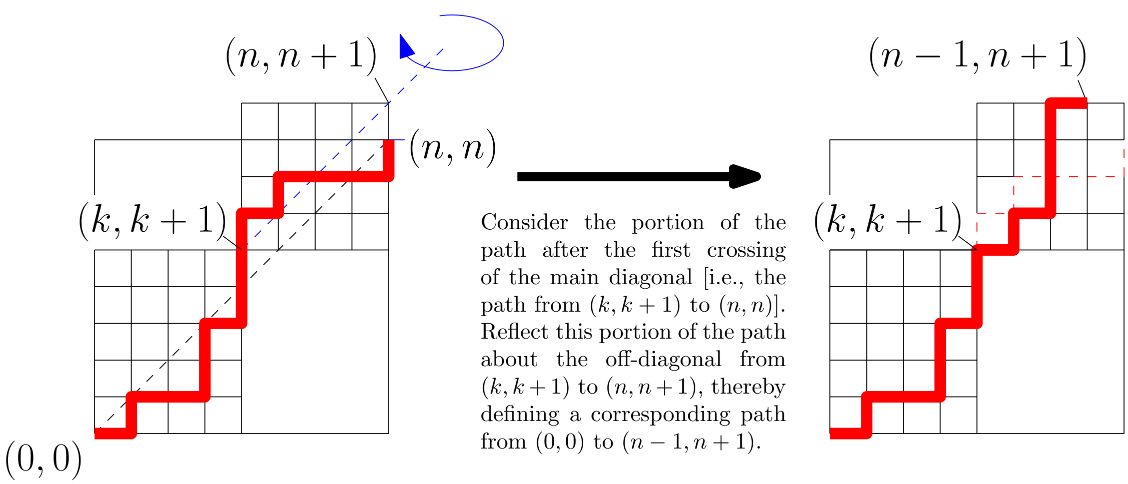
5. (a) Each path involves $2n$ total moves: n UP moves and n RIGHT moves. Different paths correspond to different sequences of the UP and RIGHT moves. The number of such paths is therefore equal to the number of different ways in which a set of n UPs and n RIGHTs can be arranged in a row of $2n$ moves; in other words, the number of ways in which, out of $2n$ move positions, n positions can be chosen for the UP moves. This is simply $\binom{2n}{n}$.

   (b) The paths that never decrease the energy to less than zero are exactly those that do not cross the diagonal from $(0, 0)$ to (n, n) (the main diagonal). In order to solve the problem, we must either count all these paths that never cross the diagonal or alternatively, count all the paths that cross the diagonal (and then subtract that count from our answer in part (a) above). There happens to be a systematic way to count all of the paths from (0,0) to (n,n) that cross the main diagonal; the key to counting these paths is to realize that we can define a one-to-one correspondence between the paths that we want to count (those which cross the diagonal) and a set of paths that is straightforward to count (as those in part (a) above).

The correspondence between the two sets of paths can best be understood visually. Consider a sample path from $(0,0)$ to (n,n) that crosses the main diagonal, as illustrated below:



For every path that crosses the main diagonal, there is a point along the path after which the mouse has completed its first step across the main diagonal. In the sample path above, this occurs at (4,5). In general, this will occur at a point $(k,k+1)$. At this point, the mouse must take an additional $n-k$ steps to the RIGHT ($\rightarrow$), and $n-k-1$ steps UP ($\uparrow$). For any such path crossing the diagonal, we can define a corresponding path from $(0,0)$ to $(n-1,n+1)$ that is defined by switching all steps that the mouse takes after its first step across the diagonal (i.e., exchange all RIGHT ($\rightarrow$) steps for UP ($\uparrow$) steps, and vice-versa). This can be visualized in the figure below, as reflecting the path after $(k,k+1)$ across the off-diagonal from $(k,k+1)$ to $(n,n+1)$:



Consider the portion of the path after the first crossing of the main diagonal [i.e., the path from $(k,k+1)$ to (n,n)]. Reflect this portion of the path about the off-diagonal from $(k,k+1)$ to $(n,n+1)$, thereby defining a corresponding path from $(0,0)$ to $(n-1,n+1)$.

It is straightforward to show that any path from $(0,0)$ to (n,n) that crosses the main diagonal has a unique corresponding path from $(0,0)$ to $(n-1,n+1)$ (found by reflecting

the original path after its first crossing of the main diagonal), and vice-versa. Note that all paths from $(0,0)$ to $(n-1,n+1)$ must cross the main diagonal (such a crossing is ensured as the path terminates on the upper side of the main diagonal), and hence the reflection back to a path from $(0,0)$ to (n,n) is well-defined. Thus we have found a one-to-one correspondence between the paths from $(0,0)$ to (n,n) that cross the main diagonal, and all paths from $(0,0)$ to $(n-1,n+1)$. Therefore, the number of paths from $(0,0)$ to (n,n) that cross the main diagonal must be equal to the number of paths from $(0,0)$ to $(n-1,n+1)$, that is, $\binom{2n}{n-1}$. It follows that the number of paths from $(0,0)$ to (n,n) that *do not* cross the main diagonal must be $\binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$ (i.e., the Catalan numbers). Hence the probability of choosing such a path is $\frac{1}{n+1}$.
[Note: This technique of counting paths that cross a barrier (in this case the main diagonal) can be generalized to a variety of problems; this general method of counting such paths is referred to as the reflection principle. Further details of counting the paths from $(0,0)$ to (n,n) that cross the main diagonal can be found, for example, under the topic "Catalan numbers" in Wikipedia.]