Recitation 6: Solutions[1]
September 23, 2008

1. (a) The possible values of the random variable X are the ten numbers $101, \ldots, 110$, and the PMF is given by

$$p_X(k) = \begin{cases} \mathbf{P}(X > k-1) - \mathbf{P}(X > k), & \text{if } k = 101, \ldots 110, \\ 0, & \text{otherwise.} \end{cases}$$

We have $\mathbf{P}(X > 100) = 1$ and for $k = 101, \ldots 110$,

$$\begin{aligned} P(X > k) &= \mathbf{P}(X_1 > k, X_2 > k, X_3 > k) \\ &= \mathbf{P}(X_1 > k)\,\mathbf{P}(X_2 > k)\,\mathbf{P}(X_3 > k) \\ &= \frac{(110-k)^3}{10^3}. \end{aligned}$$

It follows that

$$p_X(k) = \begin{cases} \dfrac{(111-k)^3 - (110-k)^3}{10^3}, & \text{if } k = 101, \ldots 110, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Since X_i is uniformly distributed over the integers in the range $[101, 110]$, we have $\mathbf{E}[X_i] = (101 + 110)/2 = 105.5$. The expected value of X is

$$\mathbf{E}[X] = \sum_{k=-\infty}^{\infty} k \cdot p_X(k) = \sum_{k=101}^{110} k \cdot p_x(k) = \sum_{k=101}^{110} k \cdot \frac{(111-k)^3 - (110-k)^3}{10^3}.$$

The above expression can be evaluated to be equal to 103.025. The expected improvement is therefore 105.5 - 103.025 = 2.475.

2. The probability of a child being a girl is $\frac{1}{2}$. Then the number of girls G out of four children is a binomial random variable. The probability mass function of G is the binomial PMF, given by

$$\begin{aligned} p_G(g) &= \binom{4}{g}\left(\frac{1}{2}\right)^g \left(1 - \frac{1}{2}\right)^{4-g} \\ &= \binom{4}{g}\left(\frac{1}{2}\right)^4, \text{ for } g = 0, 1, 2, 3, 4 \end{aligned}$$

therefore,

$$p_G(0) = \frac{1}{16}$$

[1] Published September 23, 2008

$$p_G(1) = \frac{4}{16}$$
$$p_G(2) = \frac{6}{16}$$
$$p_G(3) = \frac{4}{16}$$
$$p_G(4) = \frac{1}{16}$$

The amount of pie a daughter receives is the random variable $H = \frac{1}{G+1}$ for $G = 0, 1, 2, 3, 4$. Find the expectation of this random variable using the definition:

$$\begin{aligned} E[H] &= \sum_{g=0}^{4} \frac{1}{g+1} p_G(g) \\ &= (1/1)(\frac{1}{16}) + (1/2)(\frac{4}{16}) + (1/3)(\frac{6}{16}) + (1/4)(\frac{4}{16}) + (1/5)(\frac{1}{16}) \\ &= \frac{31}{80} \end{aligned}$$

3. Let X be a random variable representing the number of children in a randomly selected family. Then, X is geometric with parameter $p = \frac{1}{2}$. Hence, the expected number of children in the family is equal to $1/(1/2) = 2$. Since every family has exactly one girl, we have $X = 1 + B$, where B is a random variable representing the number of boys in the family. It then follows that $\mathbf{E}[B] = \mathbf{E}[X] - 1 = 1$. Therefore, the policy cannot make the expected number of girls exceed the expected number of boys. This derivation is in agreement with the argument against the hypothesis. The argument in favor of the hypothesis is flawed. The concluding clause of the argument ("Therefore, the average number of girls born must be greater than the average number of boys") is simply not implied by the fact that every family with children has a girl.

   Note that it does not matter if the number of children a given family can produce is biologically random, i.e., fertility is finite. In this case, the problem can be viewed as a "stopping rule" problem, where a family stops when the first girl is born, unless fertility runs out first. This can be modeled as a gambling problem where you receive $+1$ for every girl and -1 for every boy, and you stop when the first girl is born or fertility runs out. A famous theorem for such stopping rules, called Wald's theorem, guarantees that as long as each outcome is independent and the decision to stop is independent of the gender of possible future children, the expected win is zero, i.e. the expected numbers of girls and boys must be equal.

   As a simple example consider the case where fertility runs out after three children. The possible families and their probabilities are:

G	1/2
BG	1/4
BBG	1/8
BBB	1/8

The expected numbers of boys and girls in each family are equal:

$\mathbf{E}[G] = 1/2 + 1/4 + 1/8 = 7/8.$

$\mathbf{E}[B] = 1/4 + 2(1/8) + 3(1/8) = 7/8.$