Recitation 7: Solutions [1]
September 25, 2008

1. (a) $p_X(1) = \mathbf{P}(X=1, Y=1) + \mathbf{P}(X=1, Y=2) + \mathbf{P}(X=1, Y=3) = \frac{1}{12} + \frac{2}{12} + \frac{1}{12} = 1/3$

   (b)
$$p_{Y|X}(y \mid 1) = \frac{p_{Y,X}(y,1)}{p_X(1)} = \begin{cases} 1/4, & y=1; \\ 1/2, & y=2; \\ 1/4, & y=3; \\ 0, & \text{otherwise.} \end{cases}$$

   (c) $\mathbf{E}[Y \mid X=1] = \sum_{y=1}^{3} y\, p_{Y|X}(y \mid 1) = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} = 2$

2. (a) Let L_i be the event that Joe played the lottery on week i, and let W_i be the event that he won on week i. We are asked to find
$$\mathbf{P}(L_i \mid W_i^c) = \frac{\mathbf{P}(W_i^c \mid L_i)\mathbf{P}(L_i)}{\mathbf{P}(W_i^c \mid L_i)\mathbf{P}(L_i) + \mathbf{P}(W_i^c \mid L_i^c)\mathbf{P}(L_i^c)} = \frac{(1-q)p}{(1-q)p + 1 \cdot (1-p)} = \frac{p - pq}{1 - pq}.$$

   (b) Conditioned on X, the random variable Y is binomial:
$$p_{Y|X}(y \mid x) = \begin{cases} \binom{x}{y} q^y (1-q)^{(x-y)}, & 0 \le y \le x; \\ 0, & \text{otherwise.} \end{cases}$$

   (c) Realizing that X has a binomial PMF, we have
$$\begin{aligned} p_{X,Y}(x,y) &= p_{Y|X}(y \mid x) p_X(x) \\ &= \begin{cases} \binom{x}{y} q^y (1-q)^{(x-y)} \binom{n}{x} p^x (1-p)^{(n-x)}, & 0 \le y \le x \le n; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

   (d) Using the result from (c), we could compute
$$p_Y(y) = \sum_{x=y}^{n} p_{X,Y}(x,y) \ ,$$
   but the algebra is messy. An easier method is to realize that Y represents the number of successes in a sequence of n independent experiments, each having a probability $\mathbf{P}(W_i \mid L_i)\, \mathbf{P}(L_i) \equiv qp$ of being 1 (i.e. resulting success). Therefore Y has a binomial PMF:
$$p_Y(y) = \begin{cases} \binom{n}{y} (pq)^y (1-pq)^{(n-y)}, & 0 \le y \le n; \\ 0, & \text{otherwise.} \end{cases}$$

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(e)

$$\begin{aligned} p_{X|Y}(x \mid y) &= \frac{p_{X,Y}(x,y)}{p_Y(y)} \\ &= \begin{cases} \dfrac{\binom{x}{y} q^y(1-q)^{(x-y)} \binom{n}{x} p^x(1-p)^{(n-x)}}{\binom{n}{y} (pq)^y(1-pq)^{(n-y)}}, & 0 \le y \le x \le n; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(f) Given $Y = y$, we know that Joe played y weeks with certainty. For each of the remaining $n - y$ weeks that Joe did not win there are $x - y$ weeks where he played. Each of these events occurred with probability $\mathbf{P}(L_i \mid W_i^c)$ (the answer from part (a)). Using this logic we see that conditioned on Y, the random variable $Z = X - Y$ is binomial with $n - y$ trials and $(p - pq)\,/\,(1 - pq)$ probability of success:

$$p_{Z|Y}(z \mid y) = \begin{cases} \binom{n-y}{z} \left(\frac{p-pq}{1-pq}\right)^z \left(1 - \frac{p-pq}{1-pq}\right)^{n-y-z}, & 0 \le z \le n-y; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$p_{X|Y}(x \mid y) = \begin{cases} \binom{n-y}{x-y} \left(\frac{p-pq}{1-pq}\right)^{x-y} \left(1 - \frac{p-pq}{1-pq}\right)^{n-x}, & 0 \le y \le x \le n; \\ 0, & \text{otherwise.} \end{cases}$$

After some algebraic manipulations, the answer to (e) can be shown to be equal to the one above. Note that conditioned on Y, the random variable X is not a binomial.

3. Let X_i be the random variable taking the value 1 or 0 depending on whether the first partner of the ith couple has survived or not. Let Y_i be the corresponding random variable for the second partner of the ith couple. Then, we have $S = \sum_{i=1}^m X_i Y_i$, and by using the total expectation theorem,

$$\begin{aligned} \mathbf{E}[S \mid A = a] &= \sum_{i=1}^m \mathbf{E}[X_i Y_i \mid A = a] \\ &= m\mathbf{E}[X_1 Y_1 \mid A = a] \\ &= m\mathbf{E}[1 \times Y_1 \mid X_1 = 1,\ A = a]\,\mathbf{P}(X_1 = 1 \mid A = a) \\ &\quad + m\mathbf{E}[0 \times Y_1 \mid X_1 = 0,\ A = a]\,\mathbf{P}(X_1 = 0 \mid A = a) \\ &= m\mathbf{E}[Y_1 \mid X_1 = 1,\ A = a]\,\mathbf{P}(X_1 = 1 \mid A = a) \\ &= m\mathbf{P}(Y_1 = 1 \mid X_1 = 1,\ A = a)\mathbf{P}(X_1 = 1 \mid A = a). \end{aligned}$$

(Note that the third equality follows from the following version of the *total expectation theorem*:

$$\mathbf{E}[X \mid B] = \sum_i \mathbf{E}[X \mid A_i \cap B]\,\mathbf{P}(A_i \mid B)$$

where the events A_i form a partition of the sample space.) We have

$$\mathbf{P}(Y_1 = 1 \mid X_1 = 1,\, A = a) = \frac{a-1}{2m-1}, \qquad \mathbf{P}(X_1 = 1 \mid A = a) = \frac{a}{2m}.$$

Thus

$$\mathbf{E}\left[S \mid A = a\right] = m\,\frac{a-1}{2m-1} \cdot \frac{a}{2m} = \frac{a(a-1)}{2(2m-1)}.$$

Note that $\mathbf{E}\left[S \mid A = a\right]$ does not depend on p.