Recitation 9: Answers
October 2, 2008

1. Problem 3.1, page 184 of the text. See online solutions.

2. (a) By applying the definition of expectation, we have

$$\mathbf{E}[X] \;=\; \int x f_X(x)\,dx \;=\; \int_0^\infty x\lambda e^{-\lambda x}\,dx \;=\; \frac{1}{\lambda},$$

where the final equality can be derived using integration by parts. More generally, integration by parts can be used to prove that $\int_0^\infty x^n e^{-\lambda x}\,dx = \frac{n!}{\lambda^{n+1}}$ for any nonnegative integer n.

We can find the variance of X by evaluating

$$\mathrm{var}(X) \;=\; \mathbf{E}[X^2] - (\mathbf{E}[X])^2,$$

where

$$\mathbf{E}[X^2] \;=\; \int_0^\infty x^2\lambda e^{-\lambda x}\,dx \;=\; \frac{2}{\lambda^2}.$$

Therefore,

$$\mathrm{var}(X) \;=\; \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Finally,

$$\mathbf{P}(X \ge \mathbf{E}[X]) \;=\; \int_{\mathbf{E}[X]}^\infty f_X(x)\,dx \;=\; \int_{1/\lambda}^\infty \lambda e^{-\lambda x}\,dx \;=\; e^{-1}.$$

(b)

$$\begin{aligned}
\mathbf{P}(X > t+z \mid X > t) \;&=\; \frac{\mathbf{P}(\{X > t+z\} \cap \{X > t\})}{\mathbf{P}(X > t)} \\
&\overset{(a)}{=} \frac{\mathbf{P}(X > t+z)}{\mathbf{P}(X > t)} \\
&\overset{(b)}{=} \frac{e^{-\lambda(t+z)}}{e^{-\lambda t}} \;=\; e^{-\lambda z},
\end{aligned}$$

where (a) uses $z \ge 0$; and (b) uses $t \ge 0$ and $t+z \ge 0$. The fact that $\mathbf{P}(X > t+z \mid X > t)$ equals $\mathbf{P}(X > z)$ is called the *memoryless property* of the exponential random variable.

3. Problem 3.7, page 186 of the text. See online solutions.