Recitation 10: Answers
October 7, 2008

1. (a)

$$\begin{aligned}\int\int f_{X,Y}(x,y)\,dx\,dy &= \int_0^1\int_0^{2-2y} cxy^2\,dx\,dy \;=\; c\int_0^1 y^2\int_0^{2-2y} x\,dx\,dy \\ &= c\int_0^1 y^2\left(\tfrac{1}{2}x^2\Big|_0^{2-2y}\right)dy \;=\; c\int_0^1 y^2\left(\tfrac{1}{2}(2-2y)^2\right)dy \\ &= c\int_0^1 2y^2(1-y)^2\,dy \;=\; 2c\int_0^1\left(y^4-2y^3+y^2\right)dy \\ &= 2c\left[\frac{1}{5}y^5-\frac{1}{2}y^4+\frac{1}{3}y^3\right]\Bigg|_0^1 \;=\; \frac{1}{15}c,\end{aligned}$$

so $c = 15$.

(b) To find the marginal PDF of Y, we must integrate over the joint PDF. Note that x ranges from 0 to $2-2y$.

$$\begin{aligned}f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \\ &= \begin{cases}\int_0^{2-2y} 15xy^2\,dx, & \text{for } y\in[0,1];\\ 0, & \text{otherwise}\end{cases} \\ &= \begin{cases}30y^2(1-y)^2, & \text{for } y\in[0,1];\\ 0, & \text{otherwise}\end{cases}\end{aligned}$$

(c)

$$\begin{aligned}f_{X|Y}(x\mid y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \begin{cases}\frac{15xy^2}{30y^2(1-y)^2}, & \text{for } y\in[0,1],\ x\in[0,2-2y];\\ 0, & \text{for } y\in[0,1],\ x\notin[0,2-2y];\\ \text{undefined}, & \text{for } y\notin[0,1]\end{cases} \\ &= \begin{cases}\frac{x}{2(1-y)^2}, & \text{for } y\in[0,1],\ x\in[0,2-2y];\\ 0, & \text{for } y\in[0,1],\ x\notin[0,2-2y];\\ \text{undefined}, & \text{for } y\notin[0,1]\end{cases}\end{aligned}$$

(d) X and Y are not independent. One way to come to this conclusion is to note that the expression found for $f_{X|Y}(x\mid y)$ in the previous part has dependence on y; it would have to not depend on y for X and Y to be independent.

2. Let random variable X be the lifetime of a randomly select chip, and define events as follows:

A_t: Chip still works at time t.
B: The chip is bad.
G: The chip is good.

Using the given exponential distributions,

$$\begin{aligned} \mathbf{P}(A_t \mid G) &= \int_t^\infty \alpha e^{-\alpha x}\,dx = e^{-\alpha t} \qquad \text{and} \\ \mathbf{P}(A_t \mid B) &= \int_t^\infty 1000\alpha e^{-1000\alpha x}\,dx = e^{-1000\alpha t}. \end{aligned}$$

(a) By using the the total probability theorem,

$$\mathbf{P}(A_t) = \mathbf{P}(G)\mathbf{P}(A_t \mid G) + \mathbf{P}(B)\mathbf{P}(A_t \mid B) = pe^{-\alpha t} + (1-p)e^{-1000\alpha t}.$$

(b) We are asked for the PDF of X, and the computation of part (a) indirectly gives the CDF of X:

$$F_X(x) = \mathbf{P}(X \le x) = 1 - \mathbf{P}(A_x) = 1 - pe^{-\alpha x} - (1-p)e^{-1000\alpha x}.$$

Thus

$$f_X(x) = \frac{d}{dx}F_X(x) = p\alpha e^{-\alpha x} + (1-p)1000\alpha e^{-1000\alpha x}.$$

(c) By using the definition of conditional probability, we get:

$$\mathbf{P}(B \mid A_t) = \frac{\mathbf{P}(B \cap A_t)}{\mathbf{P}(A_t)}.$$

Furthermore, $\mathbf{P}(B \cap A_t) = \mathbf{P}(B)\mathbf{P}(A_t \mid B) = (1-p)e^{-1000\alpha t}$. Therefore,

$$\mathbf{P}(B \mid A_t) = \frac{(1-p)e^{-1000\alpha t}}{pe^{-\alpha t} + (1-p)e^{-1000\alpha t}} = \frac{1}{(p/(1-p))e^{999\alpha t} + 1}.$$

To make $\mathbf{P}(B \mid A_t) < 0.01$ when $p/(1-p) = 9$, we must have $e^{999\alpha t} > 11$, or $t > (\ln 11)/(999\alpha)$.

3. (a)

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \le y) \\ &= \mathbf{P}(X_1 \le y, X_2 \le y, \ldots, X_n \le y) \qquad \text{by meaning of maximum} \\ &= \mathbf{P}(X_1 \le y)\mathbf{P}(X_2 \le y)\cdots\mathbf{P}(X_n \le y) \qquad \text{because of independence of the } X_i\text{s} \\ &= \begin{cases} 0, & \text{for } y < 0; \\ y^n, & \text{for } y \in [0,1]; \\ 1, & \text{for } y > 1. \end{cases} \end{aligned}$$

(b)

$$\begin{aligned} F_Z(z) &= \mathbf{P}(Z \le z) \\ &= 1 - \mathbf{P}(Z > z) \\ &= 1 - \mathbf{P}(X_1 > z, X_2 > z, \ldots, X_n > z) \qquad \text{by meaning of minimum} \\ &= 1 - \mathbf{P}(X_1 > z)\mathbf{P}(X_2 > z)\cdots\mathbf{P}(X_n > z) \qquad \text{because of independence of the } X_i\text{s} \\ &= \begin{cases} 0, & \text{for } z < 0; \\ 1-(1-z)^n, & \text{for } z \in [0,1]; \\ 1, & \text{for } z > 1. \end{cases} \end{aligned}$$

(c) We wish to compute $F_{Y,Z}(y,z) = \mathbf{P}(Y \le y, Z \le z)$, and since Y and Z are not independent, we cannot simply multiply the results of the previous parts. We can, however, rely heavily on the previous parts. Using the result of part (a),

$$\begin{aligned} F_{Y,Z}(y,z) &= \mathbf{P}(Y \le y, Z \le z) \\ &= \begin{cases} 0, & \text{for } y \le 0; \\ \mathbf{P}(Z \le z \mid Y \le y)\,\mathbf{P}(Y \le y), & \text{for } y \in (0,1); \\ \mathbf{P}(Z \le z), & \text{for } y \ge 1. \end{cases} \end{aligned}$$

Note that we separate three ranges for y so that we compute $\mathbf{P}(Z \le z \mid Y \le y)$ only for values of y where the conditioning is allowed and is nontrivial. (For $y \le 0$, $\mathbf{P}(Y \le y) = 0$, so $\mathbf{P}(Z \le z \mid Y \le y)$ is undefined. For $y \ge 1$, the event $\{Y \le y\}$ always occurs, so conditioning on this event makes no difference.)

For the computation of $\mathbf{P}(Z \le z \mid Y \le y)$ with $y \in (0,1)$, note the following: Conditioning on $\{Y \le y\}$ tells us $X_i \le y$ and nothing else about X_i, since X_i is independent of X_j for every $j \ne i$. Thus conditioned on $\{Y \le y\}$, each X_i is uniform over $[0,y]$. Now following exactly the steps of part (b) gives, for $y \in (0,1)$,

$$\begin{aligned} \mathbf{P}(Z \le z \mid Y \le y) &= 1 - \mathbf{P}(Z > z \mid Y \le y) \\ &= 1 - \mathbf{P}(X_1 > z, X_2 > z, \ldots, X_n > z \mid Y \le y) \\ &= 1 - \mathbf{P}(X_1 > z \mid X_1 \le y)\mathbf{P}(X_2 > z \mid X_2 \le y)\cdots\mathbf{P}(X_n > z \mid X_n \le y) \\ &= \begin{cases} 0, & \text{for } z < 0; \\ 1 - (1 - z/y)^n, & \text{for } z \in [0,y]; \\ 1, & \text{for } z > y. \end{cases} \end{aligned}$$

Combining the results above, paying careful attention to the ranges, gives

$$F_{Y,Z}(y,z) = \begin{cases} 0, & \text{for } y \le 0 \text{ or } z < 0; \\ y^n - (y-z)^n, & \text{for } 0 \le z \le y < 1; \\ y^n, & \text{for } y \in (0,1) \text{ and } z > y; \\ 1 - (1-z)^n, & \text{for } y \ge 1 \text{ and } z \in [0,1]; \\ 1, & \text{for } y \ge 1 \text{ and } z > 1. \end{cases}$$