Recitation 11: Answers
October 9, 2008

1. Let $Y = \sqrt{|X|}$. We have, for $0 \le y \le 1$,

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}(\sqrt{|X|} \le y) = \mathbf{P}(-y^2 \le X \le y^2) = y^2,$$

(and, of course, $F_Y(y) = 0$ for $y < 0$ and $F_Y(y) = 1$ for $y > 1$). By differentiation,

$$f_Y(y) = 2y, \qquad \text{for } 0 \le y \le 1$$

(along with $f_Y(y) = 0$ for $y \notin [0, 1]$).

Let $Z = -\ln|X|$. We have, for $z \ge 0$,

$$F_Z(z) = \mathbf{P}(Z \le z) = \mathbf{P}(\ln|X| \ge -z) = \mathbf{P}(X \ge e^{-z}) + \mathbf{P}(X \le -e^{-z}) = 1 - e^{-z},$$

(along with $F_Z(z) = 0$ for $z < 0$). By differentiation,

$$f_Z(z) = e^{-z}, \qquad \text{for } z \ge 0$$

(along with $f_Z(z) = 0$ for $z < 0$).

2. Problem 3.34 from the text.

(a) Let A be the event that the first coin toss resulted in heads. To calculate the probability $\mathbf{P}(A)$, we use the continuous version of the total probability theorem:

$$\mathbf{P}(A) = \int_0^1 \mathbf{P}(A \mid X = x) f_X(x)\, dx = \int_0^1 x^2 e^x\, dx,$$

which after some calculation yields

$$\mathbf{P}(A) = e - 2.$$

(b) Using Bayes' rule,

$$\begin{aligned} f_{X|A}(x) &= \frac{\mathbf{P}(A \mid X = x) f_X(x)}{\mathbf{P}(A)} \\ &= \begin{cases} \frac{x^2 e^x}{e-2} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(c) Let B be the event that the second toss resulted in heads. We have

$$\begin{aligned} \mathbf{P}(B \mid A) &= \int_0^1 \mathbf{P}(B \mid X = x, A) f_{X|A}(x)\, dx \\ &= \int_0^1 \mathbf{P}(B \mid X = x) f_{X|A}(x)\, dx \\ &= \frac{1}{e-2} \int_0^1 x^3 e^x\, dx. \end{aligned}$$

After some calculation, this yields

$$\mathbf{P}(B \mid A) = \frac{1}{e-2} \cdot (6 - 2e) \approx \frac{0.564}{0.718} \approx 0.786.$$

3. (a) Let N_1, N_2, N_3, and N_4 denote the numbers of beeps in the four observed minutes. Because of independence of these random variables,

$$\begin{aligned}\mathbf{P}(\text{observed counts}) &= \mathbf{P}(N_1 = 1)\mathbf{P}(N_2 = 3)\mathbf{P}(N_3 = 3)\mathbf{P}(N_4 = 2) \\ &= e^{-\lambda}\frac{\lambda^1}{1!} \cdot e^{-\lambda}\frac{\lambda^3}{3!} \cdot e^{-\lambda}\frac{\lambda^3}{3!} \cdot e^{-\lambda}\frac{\lambda^2}{2!} \;=\; e^{-4\lambda}\frac{\lambda^9}{72}.\end{aligned}$$

(b) Let $g(\lambda) = \lambda^9 e^{-4\lambda}$. We wish to maximize $g(\lambda)$ over $\lambda \in (0, \infty)$. This is a simple calculus exercise.

Using the product rule of differentiation,

$$\frac{d}{d\lambda}g(\lambda) = 9\lambda^8 e^{-4\lambda} - \lambda^9 4e^{-4\lambda} = (9 - 4\lambda)\lambda^8 e^{-4\lambda},$$

so $\lambda = 9/4$ is the only critical point. Furthermore $g(9/4)$ is positive while $\lim_{\lambda\to 0} g(\lambda) = 0$ and $\lim_{\lambda\to\infty} g(\lambda) = 0$. Thus $\lambda = 9/4$ is the unique maximizer of the probability.

Instead of maximizing the probability, one can equivalently maximize the logarithm of the probability. (Since the logarithm is a strictly increasing function, the maximum is achieved at the same value of the parameter.) This often leads to slightly easier computations.

Note that the mean of a Poisson random variable with parameter λ is λ. In this problem, we have found that the maximum likelihood estimate of the parameter is the sample mean of the observations.