Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis
(Fall 2008)

Recitation 12: Solutions [1]
October 16, 2008

1. Each X_i is a binomial random variable with parameters n and $p = 1/k$.

   Let Y_i (respectively, Z_i) be a Bernoulli random variable that is equal to 1 if and only if the ith toss results in 1 (respectively, 2). We have $\mathbf{E}[Y_i Z_i] = 0$ (since $Y_i \neq 0$ implies $Z_i = 0$) and

$$\mathbf{E}[Y_i Z_j] = \mathbf{E}[Y_i]\,\mathbf{E}[Z_j] = \frac{1}{k}\cdot\frac{1}{k} \qquad \text{for} \quad i \neq j.$$

   (Y_i and Z_j are independent for $i \neq j$). Thus,

$$\begin{aligned}\mathbf{E}[X_1X_2] &= \mathbf{E}[(Y_1 + \cdots + Y_n)(Z_1 + \cdots + Z_n)] \\ &= n\mathbf{E}[Y_1(Z_1 + \cdots + Z_n)] = n(n-1)\cdot\frac{1}{k}\cdot\frac{1}{k}\end{aligned}$$

   and

$$\begin{aligned}\text{cov}(X_1, X_2) &= \mathbf{E}[X_1X_2] - \mathbf{E}[X_1]\,\mathbf{E}[X_2] \\ &= \frac{n(n-1)}{k^2} - \frac{n^2}{k^2} = -\frac{n}{k^2}.\end{aligned}$$

   The result shows that X_1 and X_2 are negatively correlated. This indicates that roughly speaking, a larger number of ones suggests a smaller number of twos. That is, the values of $X_1 - n/k$ (i.e. $X_1 - \mathbf{E}[X_1]$) and $X_2 - n/k$ ($X_2 - \mathbf{E}[X_2]$) tend to have opposite signs. The extent to which this is true, can be assessed from the magnitude of $|\rho|$ (the correlation coefficient between X_1 and X_2). Remember that the variance of a binomial random variable with parameters n and p is $np(1-p)$. We can use this fact in the computation of ρ :

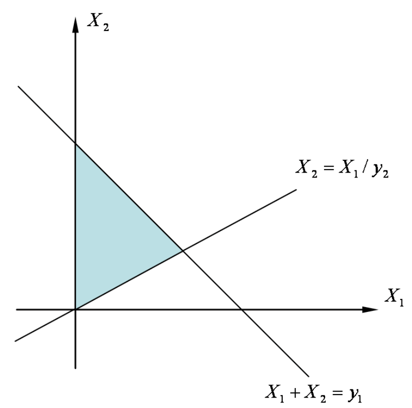
$$\rho(X_1, X_2) = \frac{\text{cov}(X_1, X_2)}{\sqrt{\text{var}(X_1)\,\text{var}(X_2)}} = \frac{-\frac{n}{k^2}}{n\frac{1}{k}\left(1 - \frac{1}{k}\right)} = \frac{-1}{k-1}.$$

   As the number k of the facets of the die grows larger, X_1 and X_2 become less correlated. Note that $\text{cov}(X_1, X_2)$ also decreases as k grows (for a fixed n). However, covariance by itself, is not an adequate measure of dependence because it can be scaled up or down by a simple scaling of the random variables without changing the relationship (to be interpreted roughly as "degree of dependence") between them. For this example, we see that the covariance grows in magnitude as n increases. However, the level of dependence between X_1 and X_2 is a function of k only, and does not grow with n.

2. We can answer this question by looking at the joint PDF of Y_1 and Y_2. Let us first find the joint CDF of Y_1 and Y_2 :

$$\begin{aligned}F_{Y_1,Y_2}(y_1, y_2) &= P(Y_1 \le y_1, Y_2 \le y_2) = P(X_1 + X_2 \le y_1, X_1/X_2 \le y_2) \\ &= P(X_2 \le y_1 - X_1, X_2 \ge X_1/y_2), \qquad y_1, y_2 > 0.\end{aligned}$$

[1] Compiled October 10, 2008



To find this probability we must integrate the joint PDF of X_1 and X_2 over the shaded area shown in the Figure.

$$\begin{aligned} F_{Y_1,Y_2}(y_1,y_2) &= \int_0^{\frac{y_1 y_2}{1+y_2}} \int_{\frac{x_1}{y_2}}^{y_1-x_1} f_{X_1,X_2}(x_1,x_2)\,dx_2 dx_1 \\ &= \int_0^{\frac{y_1 y_2}{1+y_2}} \int_{\frac{x_1}{y_2}}^{y_1-x_1} \lambda^2 e^{-\lambda(x_1+x_2)} dx_2 dx_1 \\ &= \frac{y_2 e^{-\lambda y_1}}{(1+y_2)} \left(e^{\lambda y_1} - 1 - \lambda y_1\right), \qquad y_1, y_2 \geq 0. \end{aligned}$$

To find the joint PDF of Y_1 and Y_2 :

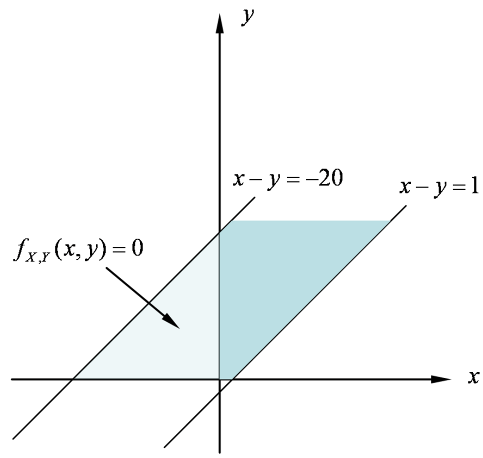
$$f_{Y_1,Y_2}(y_1,y_2) = \frac{\partial^2}{\partial y_1 \partial y_2} F_{Y_1,Y_2}(y_1,y_2) = \frac{\lambda^2 y_1 e^{-\lambda y_1}}{(y_2+1)^2} \text{ for } y_1, y_2 \geq 0.$$

Define:

$$f_{Y_1}(y_1) = \begin{cases} \lambda^2 y_1 e^{-\lambda y_1} & \text{if } \; y_1 \geq 0 \\ 0 & \text{if } \; y_1 < 0 \end{cases}, \text{ and } f_{Y_2}(y_2) = \begin{cases} \frac{1}{(y_2+1)^2} & \text{if } \; y_2 \geq 0 \\ 0 & \text{if } \; y_2 < 0 \end{cases}$$

Since $f_{Y_1,Y_2}(y_1,y_2)$ can be factorized as the product of $f_{Y_1}(y_1)$ and $f_{Y_2}(y_2)$, Y_1 and Y_2 are independent.

3. Let random variables X and Y denote the amounts of time by which Romeo and Juliet are late, and define a new random variable $Z = X - Y$. Romeo and Juliet will meet if and only if $-20 \leq Z \leq 1$.



$$\begin{aligned} P(-20 \le Z \le 1) &= \int_0^{\infty} \left(\int_{-20+y}^{1+y} f_{X,Y}(x,y)\,dx \right) dy \\ &= \int_{20}^{\infty} \left(\int_{-20+y}^{1+y} \lambda^2 e^{-\lambda(x+y)} dx \right) dy + \int_0^{20} \left(\int_0^{1+y} \lambda^2 e^{-\lambda(x+y)} dx \right) dy \\ &= 1 - \frac{1}{2} e^{-20\lambda} - \frac{1}{2} e^{-\lambda} = 0.932 \end{aligned}$$

Using the convolution formula, we can compute the PDF of the random variable $Z = X - Y$.

$$f_{X-Y}(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(x-z)\,dx$$

For $z \ge 0$, $f_Y(x-z)$ is nonzero only if $x \ge z$. Therefore:

$$\begin{aligned} f_{X-Y}(z) &= \int_0^{\infty} \lambda e^{-\lambda x} \lambda e^{-\lambda(x-z)} dx \\ &= \lambda^2 e^{\lambda z} \int_z^{\infty} e^{-2\lambda x} dx = \lambda^2 e^{\lambda z} \frac{1}{2\lambda} e^{-2\lambda z} = \frac{\lambda}{2} e^{-\lambda z}. \end{aligned}$$

Note that since X and Y are identically distributed, we can use the symmetry in the distribution of $X - Y$ and $Y - X$ to compute $f_{X-Y}(z)$ for $z < 0$:

$$f_{X-Y}(z) = f_{Y-X}(z) = f_{X-Y}(-z)\,.$$

Hence:

$$f_{Z(z)} = \begin{cases} \frac{\lambda}{2} e^{-\lambda z} & \text{if} \quad z \ge 0 \\ \frac{\lambda}{2} e^{\lambda z} & \text{if} \quad z < 0 \end{cases}$$

$$\begin{aligned} P(-20 \le Z \le 1) &= \int_{-20}^{1} f_Z(z)\,dz = \int_{-20}^{0} e^{2z} dz + \int_0^1 e^{-2z} dz \\ &= \left(\frac{1}{2} - \frac{1}{2} e^{-40} \right) + \left(\frac{1}{2} - \frac{1}{2} e^{-2} \right) = 0.932. \end{aligned}$$