Recitation 14 [1]
October 23, 2008

1. Widgets are packed into cartons which are packed into crates. The weight, X, of a widget is an exponential random variable with parameter λ :

$$f_X(x) = \lambda e^{-\lambda x}, \qquad x \geq 0.$$

The number of widgets, K, in any carton is a Poisson random variable with parameter μ :

$$p_K(k) = \frac{\mu^k e^{-\mu}}{k!}, \qquad k = 0, 1, 2, \ldots \quad .$$

The number of cartons, N, in a crate is a geometric random variable with parameter $1 - p$:

$$p_N(n) = p^{n-1}(1-p), \quad n = 1, 2, 3, \ldots \quad .$$

In the following, assume that the weights of different widgets are independent, as are the numbers of widgets in different cartons. Also assume that the number of widgets in a carton is independent of individual widget weights, and that the number of cartons in a crate is independent of how many widgets are contained in individual cartons. Determine:

(a) The probability that a randomly selected crate contains zero widgets.

(b) The expected value and variance of the number of widgets in a crate.

(c) The expected value and variance of the total weight of the widgets in a crate.

2. Define X as the height in meters of a randomly selected Alaskan, where the selection probability is equal for each Alaskan, and denote $\mathbf{E}[X]$ by h. Joe Six Packs is interested in estimating h. Because he is sure that no Alaskan is taller than 3 meters, Joe decides to use 1.5 meters as a conservative (large) value for the standard deviation of X. To estimate h, Joe averages the heights of n Alaskans that he selects at random; he denotes this quantity by H.

(a) In terms of h and Joe's 1.5 meter bound for the standard deviation of X, determine the expectation and standard deviation for H.

(b) Help Joe by calculating a minimum value of n (with $n > 0$) such that the standard deviation of Joe's estimator, H, will be less than 0.01 meters.

(c) Say Joe would like to be 99% sure that his estimate is within 5 centimeters of the true average height of Alaskans. Using the Chebyshev inequality, calculate the minimum value of n that will make Joe happy.

(d) If we agree that no Alaskan is taller than three meters, why is it correct to use 1.5 meters as an upper bound on the standard deviation for X, the height of any Alaskan selected at random?

[1] Compiled October 21, 2008

3. Random variable X is uniformly distributed between -1.0 and 1.0. Let $X_1, X_2, \ldots$, be independent identically distributed random variables with the same distribution as X. Determine which, if any, of the following sequences (all with $i = 1, 2, \ldots$) are convergent in probability. Give reasons for your answers. Include the limits if they exist.

   (a) X_i

   (b) $Y_i = \dfrac{X_i}{i}$

   (c) $Z_i = (X_i)^i$

   (d) $T_i = X_1 + X_2 + \ldots + X_i$

   (e) $U_i = \dfrac{X_1 + X_2 + \ldots + X_i}{i}$

   (f) $V_i = X_1 \cdot X_2 \cdot \ldots \cdot X_i$

   (g) $W_i = \max(X_1, \ldots, X_i)$