Recitation 14: Solutions [1]
October 23, 2008

1. (a) Let the random variable T represent the number of widgets in 1 crate and let K_i represent the number of widgets in the ith carton. Then,

$$T = K_1 + K_2 + ... + K_N$$

By the total probability theorem,

$$\begin{aligned}
\mathbf{P}(T = 0) &= \sum_{n=1}^{\infty} \mathbf{P}(T = 0|N = n)\mathbf{P}(N = n) \\
&= \sum_{n=1}^{\infty} \mathbf{P}(\{K_1 = 0\} \cap ... \cap \{K_n = 0\})\mathbf{P}(N = n) \\
&= \sum_{n=1}^{\infty} (e^{-\mu})^n \mathbf{P}(N = n) \\
&= \sum_{n=1}^{\infty} e^{-n\mu} p^{n-1}(1-p) \\
&= (1-p)e^{-\mu} \sum_{n=1}^{\infty} (pe^{-\mu})^{n-1} \\
&= (1-p)e^{-\mu} \frac{1}{1 - pe^{-\mu}}
\end{aligned}$$

(b) As above, let the random variable T represent the number of widgets in 1 crate and K_i represent the number of widgets in the ith carton. Then,

$$T = K_1 + K_2 + ... + K_N$$

Since K_i and N are all independent,

$$\mathbf{E}[T] = \mathbf{E}[K]\mathbf{E}[N] = \frac{\mu}{1-p},$$

and

$$\begin{aligned}
\text{var}(T) &= \text{var}(K)\mathbf{E}[N] + (\mathbf{E}[K])^2 \text{var}(N) \\
&= \frac{\mu}{1-p} + \frac{\mu^2 p}{(1-p)^2}.
\end{aligned}$$

(c) Let W be the total weight of the widgets in a crate: Then,

$$W = X_1 + X_2 + ... + X_T$$

Since X_i and T are all independent,

$$\mathbf{E}[W] = \mathbf{E}[X]\mathbf{E}[T] = \frac{\mu}{(1-p)\lambda},$$

[1] Compiled October 21, 2008

and

$$\begin{aligned}\text{var}(W) &= \text{var}(X)\mathbf{E}[T] + (\mathbf{E}[X])^2\text{var}(T) \\ &= \frac{1}{\lambda^2}\cdot\frac{\mu}{(1-p)} + \left(\frac{1}{\lambda}\right)^2\cdot\left(\frac{\mu}{(1-p)} + \frac{\mu^2 p}{(1-p)^2}\right) \\ &= \frac{1}{\lambda^2}\left(\frac{2\mu}{(1-p)} + \frac{\mu^2 p}{(1-p)^2}\right)\end{aligned}$$

2. Note that n is deterministic and H is a random variable.

   (a) Use X_1, X_2, ... to denote the (random) measured heights.

$$\begin{aligned}H &= \frac{X_1 + X_2 + \cdots + X_n}{n} \\ \mathbf{E}[H] &= \frac{\mathbf{E}[X_1 + X_2 + \cdots + X_n]}{n} = \frac{n\mathbf{E}[X]}{n} = h \\ \sigma_H &= \sqrt{\text{var}(H)} = \sqrt{\frac{n\ \text{var}(X)}{n^2}} \quad \text{(by independence)} \\ &\leq \frac{1.5}{\sqrt{n}}\end{aligned}$$

   (b) We solve $\frac{1.5}{\sqrt{n}} < 0.01$ for n to obtain $n > 22500$.

   (c) We want:

$$\mathbf{P}(|H - h| < 0.05) \geq 0.99$$

   Now,

$$\begin{aligned}\mathbf{P}(|H - h| < 0.05) &= \mathbf{P}(|H - E[H]| < 0.05) \\ &= 1 - \mathbf{P}(|H - E[H]| \geq 0.05) \\ &\geq 1 - \frac{\sigma_H^2}{0.05^2} \text{ (by Chebyshev inequality)} \\ &\geq 1 - \frac{1.5^2}{n0.05^2} \text{ (by part (a))}\end{aligned}$$

   Therefore, to satisfy $\mathbf{P}(|H - h| < 0.05) \geq 0.99$, we solve:

$$1 - \frac{1.5^2}{n0.05^2} \geq 0.99$$

   which gives:

$$n \geq 90000$$

(d) Intuitively, the variance of a random variable X that takes values in the range $[0, b]$ is maximum when X takes the value 0 with probability 0.5 and the value b with probability 0.5, in which case the variance of X is $b^2/4$ and its standard deviation is $b/2$.

More formally, since $\mathbf{E}[(X-c)^2]$ is minimized when $c = \mathbf{E}[X]$, we have for any random variable X taking values in $[0, b]$,

$$\begin{aligned}\text{var}(X) &\leq \mathbf{E}[(X - \frac{b}{2})^2] \\ &= \mathbf{E}[X^2] - b\mathbf{E}[X] + \frac{b^2}{4} \\ &= \mathbf{E}[X(X-b)] + \frac{b^2}{4} \\ &\leq 0 + \frac{b^2}{4},\end{aligned}$$

since $0 \leq X \leq b \Rightarrow X(X-b) \leq 0$. Thus $\sigma_X \leq b/2$.

In our example, we have $b = 3$, so $\sigma_X \leq 3/2$.

3. (a) No. Since X_i for any $i \geq 1$ is uniformly distributed between -1.0 and 1.0.

(b) Yes, to 0. Since for $\epsilon > 0$,

$$\begin{aligned}\lim_{i\to\infty} \mathbf{P}(|Y_i - 0| > \epsilon) &= \lim_{i\to\infty} \mathbf{P}(|\frac{X_i}{i} - 0| > \epsilon) \\ &= \lim_{i\to\infty} [\mathbf{P}(X_i > i\epsilon) + \mathbf{P}(X_i < -i\epsilon)] = 0.\end{aligned}$$

(c) Yes, to 0. Since for $\epsilon > 0$,

$$\begin{aligned}\lim_{i\to\infty} \mathbf{P}(|Z_i - 0| > \epsilon) &= \lim_{i\to\infty} \mathbf{P}(|(X_i)^i - 0| > \epsilon) \\ &= \lim_{i\to\infty} [\mathbf{P}(X_i > \epsilon^{\frac{1}{i}}) + \mathbf{P}(X_i < -(\epsilon)^{\frac{1}{i}})] \\ &= \lim_{i\to\infty} [\frac{1}{2}(1 - \epsilon^{\frac{1}{i}}) + \frac{1}{2}(1 - \epsilon^{\frac{1}{i}})] \\ &= 0.\end{aligned}$$

(d) No. In order for T_i to converge in probability, $T_i - T_{i-1}$ must converge to zero in probability. Since $T_i - T_{i-1} = X_i$ for all i, $T_i - T_{i-1}$ does not converge to zero, and therefore T_i does not converge in probability.

(e) Yes, to 0. Applying weak law of large numbers, we have

$$\mathbf{P}(|U_i - \mu| > \epsilon) \to 0 \text{ as } i \to \infty, \text{ for all } \epsilon > 0$$

Here $\mu = 0$ since $X_i \sim U(-1.0, 1.0)$.

(f) Yes, to 0.

$$\begin{aligned}
\mathbf{E}[V_n] &= \mathbf{E}[\mathbf{E}[V_n|X_n]] \\
&= \mathbf{E}[X_n\mathbf{E}[V_{n-1}]] = \mathbf{E}[X_n]\,\mathbf{E}[V_{n-1}] = 0 \\
\text{var}(V_n) &= \mathbf{E}[\text{var}(V_n|X_n)] + \text{var}(\mathbf{E}[V_n|X_n]) \\
&= \mathbf{E}\left[X_n^2\,\text{var}(V_{n-1})\right] + \text{var}(X_n\mathbf{E}[V_{n-1}]) \\
&= \mathbf{E}\left[X_n^2\right]\text{var}(V_{n-1}) + \mathbf{E}[V_{n-1}]^2\,\text{var}(X_n) \\
&= \frac{1}{3}\text{var}(V_{n-1}) = \left(\frac{1}{3}\right)^{n-1}\text{var}(X_1)
\end{aligned}$$

Notice that as n becomes very large, $\text{var}(V_n)$ approaches 0. By Chebyshev's inequality, we know V_n approaches $\mathbf{E}[V_n] = 0$ in probability.

(g) Yes, to 1. Since for $\epsilon > 0$,

$$\begin{aligned}
\lim_{i\to\infty}\mathbf{P}(|W_i - 1| > \epsilon) &\le \lim_{i\to\infty}\mathbf{P}(|\max\{X_1,\cdots,X_i\} - 1| > \epsilon) \\
&= \lim_{i\to\infty}[\mathbf{P}(\max\{X_1,\cdots,X_i\} > 1+\epsilon) \\
&\quad + \mathbf{P}(\max\{X_1,\cdots,X_i\} < 1-\epsilon)] \\
&= \lim_{i\to\infty}[0 + (1-\frac{\epsilon}{2})^i] \\
&= 0.
\end{aligned}$$