Recitation 15: Solutions
October 28, 2008

1. (a) No. Since X_i for any $i \geq 1$ is uniformly distributed between -1.0 and 1.0.

   (b) Yes, to 0. Since for $\epsilon > 0$,

$$\begin{aligned}\lim_{i\to\infty} \mathbf{P}(|Y_i - 0| > \epsilon) &= \lim_{i\to\infty} \mathbf{P}(|\frac{X_i}{i} - 0| > \epsilon) \\ &= \lim_{i\to\infty} [\mathbf{P}(X_i > i\epsilon) + \mathbf{P}(X_i < -i\epsilon)] = 0.\end{aligned}$$

   (c) Yes, to 0. Since for $\epsilon > 0$,

$$\begin{aligned}\lim_{i\to\infty} \mathbf{P}(|Z_i - 0| > \epsilon) &= \lim_{i\to\infty} \mathbf{P}(|(X_i)^i - 0| > \epsilon) \\ &= \lim_{i\to\infty} [\mathbf{P}(X_i > \epsilon^{\frac{1}{i}}) + \mathbf{P}(X_i < -(\epsilon)^{\frac{1}{i}})] \\ &= \lim_{i\to\infty} [\frac{1}{2}(1 - \epsilon^{\frac{1}{i}}) + \frac{1}{2}(1 - \epsilon^{\frac{1}{i}})] \\ &= 0.\end{aligned}$$

   (d) No. In order for T_i to converge in probability, $T_i - T_{i-1}$ must converge to zero in probability. Since $T_i - T_{i-1} = X_i$ for all i, $T_i - T_{i-1}$ does not converge to zero, and therefore T_i does not converge in probability.

   (e) Yes, to 0. Applying weak law of large numbers, we have

$$\mathbf{P}(|U_i - \mu| > \epsilon) \to 0 \text{ as } i \to \infty, \text{ for all } \epsilon > 0$$

   Here $\mu = 0$ since $X_i \sim U(-1.0, 1.0)$.

   (f) Yes, to 0.

$$\begin{aligned}\mathbf{E}[V_i] &= \mathbf{E}[X_1 \cdot X_2 \cdot \ldots \cdot X_i] \\ &= \mathbf{E}[X_1] \cdot \mathbf{E}[X_2] \cdot \ldots \cdot \mathbf{E}[X_i] \\ &= 0 \\ \text{var}(V_i) &= \mathbf{E}[V_i^2] - \mathbf{E}[V_i]^2 \\ &= \mathbf{E}[X_1^2 \cdot X_2^2 \cdot \ldots \cdot X_i^2] \\ &= \mathbf{E}[X_1^2] \cdot \mathbf{E}[X_2^2] \cdot \ldots \cdot \mathbf{E}[X_i^2] \\ &= \frac{2^2}{12} \cdot \frac{2^2}{12} \cdot \ldots \cdot \frac{2^2}{12} \\ &= \left(\frac{1}{3}\right)^i\end{aligned}$$

   Notice that as i becomes very large, var(V_i) approaches 0. By Chebyshev's inequality, we know V_i approaches $\mathbf{E}[V_i] = 0$ in probability.

Alternatively, we could have computed the expected value and variance of V_i, using iterated expectation and the law of total variance, respectively:

$$\begin{aligned}
\mathbf{E}[V_i] &= \mathbf{E}[\mathbf{E}[V_i|X_i]] \\
&= \mathbf{E}[X_i\mathbf{E}[V_{i-1}]] = \mathbf{E}[X_i]\mathbf{E}[V_{i-1}] = 0 \\
\mathrm{var}(V_i) &= \mathbf{E}[\mathrm{var}(V_i|X_i)] + \mathrm{var}(\mathbf{E}[V_i|X_i]) \\
&= \mathbf{E}[X_i^2\mathrm{var}(V_{i-1})] + \mathrm{var}(X_i\mathbf{E}[V_{i-1}]) \\
&= \mathbf{E}[X_i^2]\mathrm{var}(V_{i-1}) + (\mathbf{E}[V_{i-1}])^2\,\mathrm{var}(X_i) \\
&= \frac{1}{3}\mathrm{var}(V_{i-1}) = \left(\frac{1}{3}\right)^{i-1} \mathrm{var}(X_1)
\end{aligned}$$

The same conclusion would be reached.

(g) Yes, to 1. Since for $\epsilon > 0$,

$$\begin{aligned}
\lim_{i\to\infty} \mathbf{P}(|W_i - 1| > \epsilon) &\le \lim_{i\to\infty} \mathbf{P}(|\max\{X_1, \cdots, X_i\} - 1| > \epsilon) \\
&= \lim_{i\to\infty} [\mathbf{P}(\max\{X_1, \cdots, X_i\} > 1 + \epsilon) \\
&\quad + \mathbf{P}(\max\{X_1, \cdots, X_i\} < 1 - \epsilon)] \\
&= \lim_{i\to\infty} [0 + (1 - \frac{\epsilon}{2})^i] \\
&= 0.
\end{aligned}$$

2. The probability that you will believe the fair coin to be biased is the probability that the fair coin will come up with more than 525 heads out of the 1000 tosses. Let S be the number of times the coin comes up heads, which is a binomial random variable, with parameters $n = 1000$ and $p = 0.5$, so that $\mathbf{E}[S] = 1000 \cdot 0.5 = 500$ and $\sigma_S = \sqrt{1000 \cdot 0.5 \cdot 0.5} = 5\sqrt{10}$.

(a) Using the de Moivre - Laplace normal approximation to the binomial, we have

$$\begin{aligned}
\mathbf{P}(S > 525) &= \mathbf{P}(S \ge 525.5) \\
&= \mathbf{P}\left(\frac{S - 500}{5\sqrt{10}} \ge \frac{525.5 - 500}{5\sqrt{10}}\right) \\
&\approx 1 - \Phi\left(\frac{25.5}{5\sqrt{10}}\right) \\
&= 1 - \Phi(1.6128) \\
&\approx 0.0537\,.
\end{aligned}$$

(b) Using the Markov inequality, we have

$$\begin{aligned}
\mathbf{P}(S > 525) &= \mathbf{P}(S \ge 526) \\
&\le \frac{\mathbf{E}[S]}{526} \\
&= \frac{500}{526} \\
&\approx 0.951\,.
\end{aligned}$$

We see that using the Markov inequality gives us a weak upper bound, considering the approximate probability as calculated in part (a).

(c) Using the Chebyshev inequality, we have

$$\begin{aligned}
\mathbf{P}(S > 525) &= \mathbf{P}(S \geq 526) \\
&= \frac{1}{2}\left(\mathbf{P}(S \geq 526) + \mathbf{P}(S \leq 474)\right) \quad \text{(by symmetry as } p = \tfrac{1}{2}) \\
&= \frac{1}{2}\mathbf{P}(|S - 500| \geq 26) \\
&\leq \frac{1}{2}\frac{\sigma_S^2}{26^2} \\
&= \frac{1}{2}\frac{25 \cdot 10}{26^2} \\
&\approx 0.185\,.
\end{aligned}$$

We see that the Chebyshev inequality provides a substantial improvement upon the upper bound calculated by the Markov inequality in part (b).

3. (a) In this part of the problem, we need a cumulative distribution function (CDF) for the sum of 102 independent experimental values of W, the weight of a pretzel. According to a central limit theorem discussed in class, we can approximate this CDF with the CDF for a Gaussian random variable with the same expectation and variance. If we define random variable R to be the sum of 102 independent experimental values of W, we have

$$\mathbf{E}[R] = 102\mathbf{E}[W] \qquad \sigma_R^2 = 102\sigma_W^2 \quad \text{where} \quad \mathbf{P}(R \leq r) \approx \Phi\left(\frac{r - \mathbf{E}[R]}{\sigma_R}\right)$$

We can find $\mathbf{E}[W]$ and σ_W^2 using the given PDF. By inspection, $\mathbf{E}[W] = 2$. We first find $\mathbf{E}[W^2]$ to calculate $\sigma_W^2 = \mathbf{E}[W^2] - \mathbf{E}[W]^2$:

$$\mathbf{E}[W^2] \;=\; \int_0^\infty w^2 f_W(w)dw = \int_1^2 w^2(w-1)dw + \int_2^3 w^2(3-w)dw = \frac{25}{6} \Rightarrow \sigma_W^2 = \frac{1}{6}$$

So $\mathbf{E}[R] = 102 \cdot 2 = 204$ and $\sigma_R^2 = 102 \cdot \frac{1}{6} = 17$. Using the CLT approximation we have

$$\mathbf{P}(R > 200) \approx 1 - \Phi\left(\frac{200 - 204}{\sqrt{17}}\right) = 1 - \left[1 - \Phi\left(\frac{204 - 200}{\sqrt{17}}\right)\right] = \Phi(0.9701) \approx \boxed{.8340}$$

(b) We are trying to find the smallest value of n such that $\mathbf{P}(R > 200) = .990$. We will again use a CLT approximation, but this time $\mathbf{E}[R] = n\mathbf{E}[W] = 2n$ and $\sigma_R^2 = n\sigma_W^2 = \frac{n}{6}$. Since we want the weight of n pretzels to exceed 200 ounces with probability .990, we have

$$1 - \mathbf{P}(R \leq 200) = 0.990 \quad \Rightarrow \quad 1 - \Phi\left(\frac{200 - 2n}{\sqrt{\frac{n}{6}}}\right) \approx .990 \quad \Rightarrow \quad \Phi\left(\frac{200 - 2n}{\sqrt{\frac{n}{6}}}\right) \approx .01$$

When $\Phi(y_0) < 0.5$, then y_0 is negative. Furthermore, because $\Phi(-|y_0|) = 1 - \Phi(|y_0|)$, we have

$$1 - \Phi\left(\frac{-200 + 2n}{\sqrt{\frac{n}{6}}}\right) \approx .01 \quad \Rightarrow \quad \Phi\left(\frac{-200 + 2n}{\sqrt{\frac{n}{6}}}\right) \approx .990 \quad \Rightarrow \quad \frac{-200 + 2n}{\sqrt{\frac{n}{6}}} \approx 2.33$$

So,

$$\begin{aligned} 40000 - 800n + 4n^2 \approx \frac{2.33^2}{6}n \quad &\Rightarrow \quad 4n^2 - 800.905n + 40000 \approx 0 \\ &\Rightarrow \quad n \approx 104.87 \text{ or } 95.36 \end{aligned}$$

Which value is correct? Consider $\mathbf{E}[R]$ for each possible value of n. $\mathbf{E}[R] = 2n$ must be greater than 200 for the question to make sense, as we showed above that $y_0 = (200 - 2n)/\sigma_R < 0$. The value that achieves this is $n = 104.87$. This number corresponds to the amount of pretzels we need so that their total weight is 200 ounces with probability .990. Therefore, to find the *smallest integer* for which the total weight *exceeds* 200 ounces with probability .990, we round *up* to $\boxed{105.}$