

Recitation 16: Solutions
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1. (a) Failed quizzes are a Bernoulli process with parameter $p = 1/4$. The desired probability is given by the binomial formula:

$$\binom{6}{2} p^2 (1-p)^4 = \frac{6!}{4!2!} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^4.$$

- (b) The expected number of quizzes up to the third failure is the expected value of a Pascal random variable of order three, with parameter $1/4$, which is $3 \cdot 4 = 12$. Subtracting the number of failures, we have that the expected number of quizzes that Dave will pass is $12 - 3 = 9$.

- (c) The event of interest is the intersection of the following three independent events:

A: there is exactly one failure in the first seven quizzes.

B: quiz eight is a failure.

C: quiz nine is a failure.

We have

$$\mathbf{P}(A) = \binom{7}{1} \left(\frac{1}{4}\right) \left(\frac{3}{4}\right)^6, \quad \mathbf{P}(B) = \mathbf{P}(C) = \frac{1}{4},$$

so the desired probability is

$$\mathbf{P}(A \cap B \cap C) = 7 \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^6.$$

- (d) Let *B* be the event that Dave fails two quizzes in a row before he passes two quizzes in a row. Let us use *F* and *S* to indicate quizzes that he has failed or passed, respectively. We then have

$$\begin{aligned} \mathbf{P}(B) &= \mathbf{P}(FF \cup SFF \cup FSFF \cup SFSFF \cup FSFSFF \cup SFSFSFF \cup \dots) \\ &= \mathbf{P}(FF) + \mathbf{P}(SFF) + \mathbf{P}(FSFF) + \mathbf{P}(SFSFF) + \mathbf{P}(FSFSFF) \\ &\quad + \mathbf{P}(SFSFSFF) + \dots \\ &= \left(\frac{1}{4}\right)^2 + \frac{3}{4} \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \frac{3}{4} \left(\frac{1}{4}\right)^2 + \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} \left(\frac{1}{4}\right)^2 \\ &\quad + \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} \left(\frac{1}{4}\right)^2 + \dots \\ &= \left[\left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \frac{3}{4} \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} \left(\frac{1}{4}\right)^2 + \dots \right] \\ &\quad + \left[\frac{3}{4} \left(\frac{1}{4}\right)^2 + \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} \left(\frac{1}{4}\right)^2 + \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} \left(\frac{1}{4}\right)^2 + \dots \right]. \end{aligned}$$

Therefore, $\mathbf{P}(B)$ is the sum of two infinite geometric series, and

$$\mathbf{P}(B) = \frac{\left(\frac{1}{4}\right)^2}{1 - \frac{1}{4} \cdot \frac{3}{4}} + \frac{\frac{3}{4} \left(\frac{1}{4}\right)^2}{1 - \frac{3}{4} \cdot \frac{1}{4}} = \frac{7}{52}.$$

Alternatively, the problem can be solved with repeated application of the law of total probability. Let F_i be the event that Dave fails the i th quiz. From the law of total probability,

$$\begin{aligned} \mathbf{P}(B) &= \mathbf{P}(B|F_1)\mathbf{P}(F_1) + \mathbf{P}(B|F_1^c)\mathbf{P}(F_1^c) \\ &= \frac{1}{4}\mathbf{P}(B|F_1) + \frac{3}{4}\mathbf{P}(B|F_1^c), \end{aligned}$$

which expresses $\mathbf{P}(B)$ in terms of two conditional probabilities, $\mathbf{P}(B|F_1)$ and $\mathbf{P}(B|F_1^c)$. We first focus on the first one of these conditional probabilities and apply the law of total probability:

$$\begin{aligned} \mathbf{P}(B|F_1) &= \mathbf{P}(B|F_1 \cap F_2)\mathbf{P}(F_2|F_1) + \mathbf{P}(B|F_1 \cap F_2^c)\mathbf{P}(F_2^c|F_1) \\ &= \frac{1}{4} + \frac{3}{4}\mathbf{P}(B|F_1^c), \end{aligned}$$

where the final equality holds because the probability of Dave failing two consecutive quizzes only depends on his last quiz's performance (provided he has not yet failed or passed two in a row). We now have an equation involving these two conditional probabilities. By again applying the law of total probability to $\mathbf{P}(B|F_1)$ with the partition $F_2, F_2^c \cap F_3, F_2^c \cap F_3^c$, we can solve for one of them,

$$\begin{aligned} \mathbf{P}(B|F_1) &= \mathbf{P}(B|F_1 \cap F_2)\mathbf{P}(F_2|F_1) + \mathbf{P}(B|F_1 \cap F_2^c \cap F_3)\mathbf{P}(F_2^c \cap F_3|F_1) \\ &\quad + \mathbf{P}(B|F_1 \cap F_2^c \cap F_3^c)\mathbf{P}(F_2^c \cap F_3^c|F_1) \\ &= \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4}\mathbf{P}(B|F_1) \\ \mathbf{P}(B|F_1) &= \frac{4}{13}. \end{aligned}$$

By substituting our value of $\mathbf{P}(B|F_1)$ into the equation above that also involves $\mathbf{P}(B|F_1^c)$, we find that $\mathbf{P}(B|F_1^c) = \frac{1}{13}$. Lastly, by substituting the values of both conditional probabilities into our first equation, we find,

$$\begin{aligned} \mathbf{P}(B) &= \frac{1}{4} \frac{4}{13} + \frac{3}{4} \frac{1}{13} \\ &= \frac{7}{52}, \end{aligned}$$

which concurs with the solution via geometric series.

- (e) Quizzes for which Dave receives a candy bar are a split Bernoulli process with parameter $p = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$. The expected value of the k th arrival time of a Bernoulli process is $\frac{k}{p}$, and hence the expected number of quizzes that Dave must take in order to receive 6 candy bars is $\frac{6}{\frac{3}{8}} = 16$.
- (f) If we only consider the sequence of quizzes which Dave passes, the quizzes for which Dave receives a candy bar are a Bernoulli process with parameter $p = \frac{1}{2}$. The expected value of the k th arrival time of a Bernoulli process is $\frac{k}{p}$, and hence the expected number of quizzes that Dave must pass in order to receive 6 candy bars is $\frac{6}{\frac{1}{2}} = 12$.

2. Problem 6.4, page 327 of the text. See solution in text.

3. **Random incidence in the Bernoulli process.** Problem 6.5, pages 327-328 in the text. See solution in the text.