Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2008)

Recitation 18: Solutions

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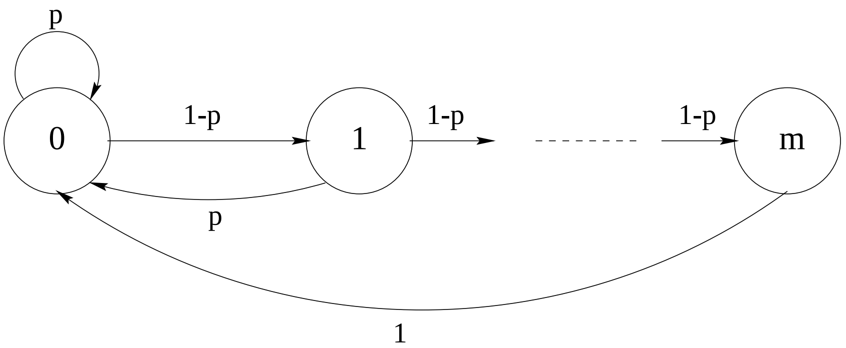
1. (a) This Markov chain will have a steady state distribution when $0 < f < 1$, as this will make the Markov chain a single recurrent class which is aperiodic (since the recurrent class includes states with non-zero self-transition probabilities). We can solve for the steady state probabilities, π_1, π_2, and π_3, using the steady state convergence theorem.

   The balance equations and normalization equation state

$$\begin{aligned} \pi_1 &= \pi_1 \cdot 0 + \pi_2 \cdot 0 + \pi_3 \cdot f \\ &= \pi_3 \cdot f \\ \pi_2 &= \pi_1 \cdot 1 + \pi_2 \cdot (1-f) + \pi_3 \cdot 0 \\ &= \pi_3 \cdot f + \pi_2 \cdot (1-f) \\ &= \pi_3 \\ 1 &= \pi_1 + \pi_2 + \pi_3 \\ &= \pi_3 \cdot f + \pi_3 + \pi_3 \\ &= \pi_3 \cdot (2+f) \end{aligned}$$

   Therefore, $\pi_2 = \pi_3 = 1/(2+f)$, and $\pi_1 = f/(2+f)$.

2. We introduce the states $0, 1, \ldots, m$ and identify them as the number of days the gate survives a crash. The state transition diagram is shown in the figure below.



The balance equations take the form,

$$\begin{aligned} \pi_0 &= \pi_0 p + \pi_1 p + \cdots + \pi_{m-1} p + \pi_m \\ \pi_1 &= \pi_0(1-p) \\ \pi_2 &= \pi_1(1-p) = \pi_0(1-p)^2 \\ &\vdots \\ \pi_m &= \pi_0(1-p)^m \end{aligned}$$

These equations together with the normalization equation have a unique solution which gives us the steady-state probabilities of all states. The steady-state expected frequency of gate replacements is the expected frequency of visits to state 0, which by frequency interpretation is given by π_0. Solving the above equations with the normalization equation we get,

$$\begin{aligned} E[\text{frequency of gate replacements}] &= \pi_0 \\ &= \frac{p}{1-(1-p)^{m+1}} \end{aligned}$$

3. (a) The local balance equations take the form

$$0.6\pi_1 = 0.3\pi_2, \qquad 0.2\pi_2 = 0.2\pi_3.$$

They can be solved, together with the normalization equation, to yield

$$\pi_1 = \frac{1}{5}, \qquad \pi_2 = \pi_3 = \frac{2}{5}.$$

(b) The probability that the first transition is a birth is

$$0.6\pi_1 + 0.2\pi_2 = \frac{0.6}{5} + \frac{0.2 \cdot 2}{5} = \frac{1}{5}.$$

(c) If the state is 1, which happens with probability 1/5, the first change of state is certain to be a birth. If the state is 2, which happens with probability 2/5, the probability that the first change of state is a birth is equal to $0.2/(0.3+0.2) = 2/5$. Finally, if the state is 3, the probability that the first change of state is a birth is equal to 0. Thus, the probability that the first change of state that we observe is a birth is equal to

$$1 \cdot \frac{1}{5} + \frac{2}{5} \cdot \frac{2}{5} = \frac{9}{25}.$$

(d) We have

$$\begin{aligned} \mathbf{P}(\text{state was 2} \mid \text{first transition is a birth}) &= \frac{\mathbf{P}(\text{state was 2 and first transition is a birth})}{\mathbf{P}(\text{first transition is a birth})} \\ &= \frac{\pi_2 \cdot 0.2}{1/5} = \frac{2}{5}. \end{aligned}$$

(e) As shown in part (c), the probability that the first change of state is a birth is 9/25. Furthermore, the probability that the state is 2 and the first change of state is a birth is $2\pi_2/5 = 4/25$. Therefore, the desired probability is

$$\frac{4/25}{9/25} = \frac{4}{9}.$$

(f) In a birth-death process, there must be as many births as there are deaths, plus or minus 1. Thus, the steady-state probability of births must be equal to the steady-state probability of deaths. Hence, in steady-state, half of the state changes are expected to be births. Therefore, the conditional probability that the first observed transition is a birth, given that it resulted in a change of state, is equal to 1/2. This answer can also be obtained algebraically:

$$\mathbf{P}(\text{birth} \mid \text{change of state}) = \frac{\mathbf{P}(\text{birth})}{\mathbf{P}(\text{change of state})} = \frac{1/5}{\frac{1}{5}\cdot 0.6 + \frac{2}{5}\cdot 0.5 + \frac{2}{5}\cdot 0.2} = \frac{1/5}{2/5} = \frac{1}{2}.$$

(g) We have

$$\mathbf{P}(\text{leads to state 2} \mid \text{change}) = \frac{\mathbf{P}(\text{change that leads to state 2})}{\mathbf{P}(\text{change})} = \frac{\pi_1 \cdot 0.6 + \pi_3 \cdot 0.2}{2/5} = \frac{1}{2}.$$

This is intuitive because for every change of state that leads into state 2, there must be a subsequent change of state that leads away from state 2.