Recitation 19 - Solutions
November 18, 2008

1. (a) The prior PDF of Θ is

$$f_\Theta(\theta) = \begin{cases} 1 & \text{if } 0 \le \theta \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and the conditional PDF of the observation is

$$f_{X|\Theta}(x \mid \theta) = \begin{cases} \frac{1}{\theta} & \text{if } 0 \le x \le \theta \\ 0 & \text{otherwise} \end{cases}$$

We can easily compute

$$f_{\Theta,X}(\theta, x) = f_\Theta(\theta) f_{X|\Theta}(x \mid \theta) = \begin{cases} \frac{1}{\theta} & \text{if } 0 < x \le \theta \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_X(x) = \int_0^1 f_\Theta(\theta) f_{X|\Theta}(x \mid \theta)\, d\theta = \int_x^1 \frac{1}{\theta}\, d\theta = \ln \frac{1}{x}.$$

Using Bayes' rule, we obtain the posterior PDF of Θ

$$f_{\Theta|X}(\theta \mid x) = \frac{f_\Theta(\theta) f_{X|\Theta}(x \mid \theta)}{f_X(x)} = \begin{cases} \frac{1}{\theta \ln \frac{1}{x}} & \text{if } 0 < x \le \theta \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Similar to the case when $n = 1$, we have

$$f_{X_1,\ldots,X_n|\Theta}(x_1, \ldots, x_n \mid \theta) = \begin{cases} \frac{1}{\theta^n} & \text{if } 0 < x_{\max} \le \theta \le 1 \\ 0 & \text{otherwise} \end{cases}$$

where

$$x_{\max} = \max\{x_1, \cdots, x_n\}.$$

We can now obtain the joint distribution of $X_1, \ldots, X_n$ for $n > 1$:

$$\begin{aligned} f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) &= \int_0^1 f_\Theta(\theta) f_{X_1,\ldots,X_n|\Theta}(x_1, \ldots, x_n \mid \theta)\, d\theta = \int_{x_{\max}}^1 \frac{1}{\theta^n}\, d\theta \\ &= \begin{cases} \frac{\frac{1}{x_{\max}^{n-1}} - 1}{n-1} & \text{if } 0 < x_{\max} \le 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The posterior PDF of Θ is therefore

$$f_{\Theta|X_1,\ldots,X_n}(\theta \mid x_1, \ldots, x_n) = \begin{cases} \frac{c(x_{\max})}{\theta^n} & \text{if } 0 < x_{\max} \le \theta \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Where $c(x_{\max}) = \frac{n-1}{\frac{1}{x_{\max}^{n-1}} - 1}$ is a normalizing constant that depends only on $x_{\max}$.

2. (a) From Bayes' rule, the posterior PDF of Θ has the form

$$f_{\Theta|X}(\theta \mid x) = \begin{cases} c(n,x)\, f_\Theta(\theta)\, \theta^x\, (1-\theta)^{n-x} & \text{if } 0 < \theta < 1 \\ 0 & \text{otherwise} \end{cases}$$

where $c(n,x)$ is a normalizing constant (independent of θ).

(b) Substituting, we have the posterior PDF of Θ

$$f_{\Theta|X}(\theta \mid k) == \begin{cases} \frac{\theta^{\alpha+x}(1-\theta)^{\beta+n-x}}{C(\alpha,\beta,n,x)} 0 < \theta < 1 & \\ 0 & \text{otherwise} \end{cases},$$

where $C(\alpha, \beta, n, x)$ is again the normalization constant. The posterior PDF is also a beta density, with parameters $\alpha' = \alpha + x$ and $\beta' = \beta + n - k$.

The graph below illustrates the prior for $\alpha = 1, \beta = 1$ and the resulting posterior for $n = 8$ and $x = 1, 3, 4$: