Recitation 20 - Solutions
November 20, 2008

1. (a) We will decide the alternative hypothesis is true if

$$\begin{aligned} f_{X|\Theta}(x \mid 1)p_\Theta(1) &\geq f_{X|\Theta}(x \mid 0)p_\Theta(0) \\ 2x \cdot p &\geq 1 \cdot (1-p) \\ x &\geq \frac{1-p}{2p}, \quad \text{for } x \in [0,1]. \end{aligned}$$

If $p = 1/2$, the rule above corresponds to $x \geq 1/2$.
If $p = 2/3$, the rule above corresponds to $x \geq 1/4$.
If $p = 1/3$, the rule above corresponds to $x \geq 1$. In other words, we will always decide that the null hypothesis is true. Since the threshold is a monotonic function of p, this is true for any $p \leq 1/3$.

(b) If the null hypothesis is true, the error occurs when we decide the alternative hypothesis was true. For $p = 2/3$, this corresponds to the event $\{X \geq 1/4\}$. Therefore,

$$\mathbf{P}(\text{error} \mid \Theta = 0) = \mathbf{P}(X \geq 1/4 \mid \Theta = 0) = \int_{1/4}^{\infty} f_{X|\Theta}(x \mid 0)\,dx = \int_{1/4}^{1} 1\,dx = 3/4.$$

(c) Similar to the computation above, we find for $p > 1/3$

$$\mathbf{P}(\text{error} \mid \Theta = 0) = \mathbf{P}\left(X \geq \frac{1-p}{2p} \mid \Theta = 0\right) = \int_{\frac{1-p}{2p}}^{\infty} f_{X|\Theta}(x \mid 0)\,dx = \int_{\frac{1-p}{2p}}^{1} 1\,dx = 1-\frac{1-p}{2p} = \frac{3p-1}{2p}.$$

$$\mathbf{P}(\text{error} \mid \Theta = 1) = \mathbf{P}\left(X < \frac{1-p}{2p} \mid \Theta = 1\right) = \int_{-\infty}^{\frac{1-p}{2p}} f_{X|\Theta}(x \mid 1)\,dx = \int_{0}^{\frac{1-p}{2p}} 2x\,dx = \left(\frac{1-p}{2p}\right)^2.$$

Now using the total probability law, we find

$$\begin{aligned} \mathbf{P}(\text{error}) &= \mathbf{P}(\text{error} \mid \Theta = 0)p_\Theta(0) + \mathbf{P}(\text{error} \mid \Theta = 1)p_\Theta(1) \\ &= \frac{(3p-1)(1-p)}{2p} + \frac{(1-p)^2}{4p} = \frac{(1-p)(5p-1)}{4p}. \end{aligned}$$

For $p \leq 1/3$, we will always decide on the null hypothesis, and the resulting probability of error is

$$\mathbf{P}(\text{error}) = \mathbf{P}(\text{error} \mid \Theta = 0)p_\Theta(0) + \mathbf{P}(\text{error} \mid \Theta = 1)p_\Theta(1) = 0 \cdot (1-p) + 1 \cdot p = p.$$

For the boundary value of $p = 1/3$, both formulas yield $\mathbf{P}(\text{error}) = 1/3$.

2. (a) We first find the posterior distribution for Θ:

$$\begin{aligned}
f_{\Theta|X}(\theta \mid k) &= \frac{p_{X|\Theta}(k \mid \theta) \cdot f_\Theta(\theta)}{p_X(k)} \\
&= \frac{p_{X|\Theta}(k \mid \theta) \cdot f_\Theta(\theta)}{\int_0^1 p_{X|\Theta}(k \mid t) \cdot f_\Theta(t) dt} \\
&= \frac{\binom{n}{k} \cdot \theta^k \cdot (1-\theta)^{(n-k)} \cdot 1}{\int_0^1 \binom{n}{k} \cdot t^k \cdot (1-t)^{(n-k)} \cdot 1 dt} \\
&= \frac{\binom{n}{k} \cdot \theta^k \cdot (1-\theta)^{(n-k)}}{\binom{n}{k} \int_0^1 \cdot t^k \cdot (1-t)^{(n-k)} dt} \\
&= \frac{\theta^k \cdot (1-\theta)^{(n-k)}}{\frac{k!(n-k)!}{(k+n-k+1)!}} \\
&= \frac{(n+1)!}{k!(n-k)!}\, \theta^k \,(1-\theta)^{(n-k)}
\end{aligned}$$

To find the MAP estimate, we need to find the value $\hat{\theta}$ that maximizes the posterior. We differentiate the posterior PDF and set the derivative to 0 then solve for θ, obtaining,

$$k\theta^{k-1}(1-\theta)^{n-k} - (n-k)\theta^k(1-\theta)^{n-k-1} = 0$$

which yields

$$\hat{\theta}_{\mathrm{MAP}}(k) = \frac{k}{n}.$$

(b) The conditional mean estimate can be found by integrating

$$\begin{aligned}
\mathbf{E}[\Theta \mid X = k] &= \int_0^1 \theta \frac{(n+1)!}{k!(n-k)!}\, \theta^k \,(1-\theta)^{(n-k)}\, d\theta \\
&= \frac{(n+1)}{k!(n-k)!} \int_0^1 \theta^{k+1} \cdot (1-\theta)^{(n-k)} d\theta \\
&= \frac{(n+1)}{k!(n-k)!} \cdot \frac{(k+1)!(n-k)!}{(n+2)!} \\
&= \frac{k+1}{n+2}
\end{aligned}$$

and

$$\hat{\theta}_{\mathrm{CE}}(k) = \mathbf{E}[\Theta \mid X = k] = \frac{k+1}{n+2}.$$

(c) The conditional mean estimator adjusts the counts of both heads and tails by 1 each in the formula above.

(d) We recompute the conditional distribution using this new prior

$$\begin{aligned}
f_{\Theta|X}(\theta \mid k) &= \frac{p_{X|\Theta}(k \mid \theta)\cdot f_\Theta(\theta)}{p_X(k)} \\
&= \frac{p_{X|\Theta}(k \mid \theta)\cdot f_\Theta(\theta)}{\int_0^1 p_{X|\Theta}(k \mid t)\cdot f_\Theta(t)dt} \\
&= \frac{\binom{n}{k}\cdot\theta^k\cdot(1-\theta)^{(n-k)}\cdot\frac{\theta^\alpha(1-\theta)^\beta}{B(\alpha,\beta)}}{\int_0^1\binom{n}{k}\cdot t^k\cdot(1-t)^{(n-k)}\cdot\frac{t^\alpha(1-t)^\beta}{B(\alpha,\beta)}dt} \\
&= \frac{\binom{n}{k}\cdot\frac{1}{B(\alpha,\beta)}\cdot\theta^{k+\alpha}\cdot(1-\theta)^{(n-k)+\beta}}{\binom{n}{k}\cdot\frac{1}{B(\alpha,\beta)}\cdot\int_0^1 t^{k+\alpha}\cdot(1-t)^{(n-k)+\beta}dt} \\
&= \frac{\theta^{k+\alpha}\cdot(1-\theta)^{(n-k)+\beta}}{\int_0^1 t^{k+\alpha}\cdot(1-t)^{(n-k)+\beta}dt} \\
&= \frac{(n+1+\alpha+\beta)!}{(k+\alpha)!(n-k+\beta)!}\,\theta^{k+\alpha}\,(1-\theta)^{(n-k)+\beta}
\end{aligned}$$

To find the MAP estimate, we differentiate the posterior PDF and set the derivative to 0 then solve for θ, obtaining,

$$(k+\alpha)\theta^{k+\alpha-1}(1-\theta)^{n-k+\beta}-(n-k+\beta)\theta^{k+\alpha}(1-\theta)^{n-k+\beta-1}=0$$

which yields

$$\hat{\theta}_{\text{MAP}}(k)=\frac{k+\alpha}{n+\alpha+\beta}$$

and we can compute the conditional expectation similarly as we had done before,

$$\begin{aligned}
\mathbf{E}[\Theta \mid X=k] &= \int_0^1 \theta\cdot\theta^{k+\alpha}\cdot(1-\theta)^{(n-k)+\beta}\cdot\frac{(n+1+\alpha+\beta)!}{(k+\alpha)!(n-k+\beta)!}d\theta \\
&= \frac{(n+1+\alpha+\beta)!}{(k+\alpha)!(n-k+\beta)!}\int_0^1\theta^{k+\alpha+1}\cdot(1-\theta)^{(n-k)+\beta}d\theta \\
&= \frac{(n+1+\alpha+\beta)!}{(k+\alpha)!(n-k+\beta)!}\cdot\frac{(k+\alpha+1)!(n-k+\beta)!}{(n+\alpha+\beta+2)!} \\
&= \frac{k+\alpha+1}{n+\alpha+\beta+2}
\end{aligned}$$

and

$$\hat{\theta}_{\text{CE}}(k)=\mathbf{E}[\Theta \mid X=k]=\frac{k+\alpha+1}{n+\alpha+\beta+2}.$$

Note how the prior affects the estimators.