MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2008)

Recitation 21 - Solutions
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1. Let Θ be the car speed and let X be the radar's measurement. The joint PDF of Θ and X is uniform over the set of pairs (θ, x) that satisfy $55 \leq \theta \leq 75$ and $\theta \leq x \leq \theta + 5$.

   For any given x, the value of Θ is constrained to lie on a particular interval, the posterior PDF of Θ is uniform over that interval, and the conditional mean is the midpoint of that interval. In particular,

$$\mathbf{E}[\Theta \mid X = x] = \begin{cases} \dfrac{x}{2} + 27.5, & \text{if } 55 \leq x \leq 60, \\ x - 2.5, & \text{if } 60 \leq x \leq 75, \\ \dfrac{x}{2} + 35, & \text{if } 75 \leq x \leq 80. \end{cases}$$

2. (a) hypothesis testing

   (b) estimation

   (c) The LMS (Least Mean Squares) estimator, $\hat{A}$, is given by $E[A|K = 2]$. We first calculate the posterior PDF $f_{A|K}(a|2)$. From Bayes' rule , the posterior PDF of M has the following form,

$$f_{A|K}(a|2) \quad = \quad \frac{P_{K|A}(2|a) f_A(a)}{P_K(2)}.$$

   $P_{K|A}(2|a)$ is a binomial distribution with parameters $n = 3$ and $p = a$. Therefore,

$$P_{K|A}(2|a) = 3a^2(1 - a).$$

   The probability , $P_K(2)$, can be derived through the total probability theorem, in particular,

$$\begin{aligned} P_K(2) &= \int_0^1 P_{K|A}(2|a) f_A(a) da, \\ &= \int_0^1 3a^2(1 - a) f_A(a) da, \\ &= \frac{1}{4}. \end{aligned}$$

   The LMS (Least Mean Squares), $\hat{A}$, is therefore given by,

$$\begin{aligned} E[A|K = 2] &= \int_0^1 a f_{A|K}(a|2) da, \\ &= \int_0^1 12a^3(1 - a) da, \\ &= 0.6. \end{aligned}$$

   (d) You would expect the conditional mean, given $K = 2$, to move down toward 1/2 because the new density is more centered (i.e., has a lower standard deviation) about the same mean (= 0.5) as the previous one.

We proceed in a similar manner to (c). Since $f_M(m)$ has changed, the probability , $P_K(2)$, is now given by,

$$\begin{aligned} P_K(2) &= \int_0^1 P_{K|A}(2|a) f_A(a) da, \\ &= \int_0^1 18a^3(1-a)^2 f_A(a) da, \\ &= \frac{3}{10}. \end{aligned}$$

The new LMS (Least Mean Squares) estimator, $\hat{A}$, is given by,

$$\begin{aligned} E[A|K=2] &= \int_0^1 a f_{A|K}(a|2) da, \\ &= \int_0^1 60a^4(1-a) da, \\ &= \frac{12}{21} = 0.571. \end{aligned}$$

3. (a) In the absence of any observation, $\mathbf{E}\left[(\Theta-\hat{\theta})^2\right]$ is minimized when $\hat{\theta} = \mathbf{E}[\Theta]$. Since the distribution of θ is symmetric around its mean, $\hat{\theta} = 2$. Alternatively:

$$\hat{\theta} = \mathbf{E}[\Theta] = \int_1^2 \theta(\theta-1)\, d\theta + \int_2^3 \theta(3-\theta)\, d\theta = \frac{5}{6} + \frac{7}{6} = 2$$

(b) We need to compute $f_{\Theta,X}(\theta, x)$:

$$f_{\Theta,X}(\theta,x) = f_\Theta(\theta) f_{X|\Theta}(x|\theta)$$

Let us first compute $f_{W|\Theta}(w|\theta)$:

$$f_{W|\Theta}(w|\theta) = \frac{f_{W,\Theta}(w,\theta)}{f_\Theta(\theta)} = \begin{cases} 0.5/(\theta-1) & \text{if } 1 < \theta \le 2, \ -2\theta+3 \le w \le 1 \\ 0.5/(3-\theta) & \text{if } 2 \le \theta < 3, \ -1 \le w \le -2\theta+5 \\ \text{undefined} & \text{if } \theta \le 1, \text{ or } \theta \ge 3. \\ 0 & , \text{ otherwise.} \end{cases}$$

Hence:

$$f_{X|\Theta}(x|\theta) = \begin{cases} 0.5/(\theta-1) & \text{if } 1 < \theta \le 2, \ -\theta+3 \le x \le 1+\theta \\ 0.5/(3-\theta) & \text{if } 2 \le \theta < 3, \ -1+\theta \le x \le -\theta+5 \\ \text{undefined} & \text{if } \theta \le 1, \text{ or } \theta \ge 3. \\ 0 & , \text{ otherwise.} \end{cases}$$

we have:

$$f_{\Theta,X}(\theta,x) = f_\Theta(\theta) f_{X|\Theta}(x|\theta) = \begin{cases} 0.5 & , \ |x-2| + |\theta-2| \le 1 \\ 0 & , \ \text{otherwise} \end{cases}$$

The region over which the joint distribution is nonzero is shown in Figure 2. Over this region, the joint distribution is uniform. Hence, the LMS estimator equals the midpoint of the density footprint. That is:

$$\hat{\theta} = \mathbf{E}[\Theta \mid X = x] = 2$$

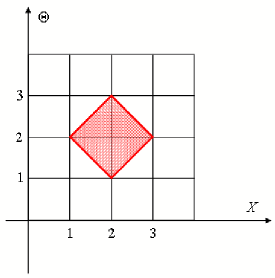

Figure 1: The joint distribution of Θ and X is uniform over the shaded area.

(This can be verified analytically as well). Hence, $\hat{\theta}$, the LMS estimate of Θ is the same in the presence or absence of observation of X. In other words, observing the value of X does not change our estimate. Does this mean that observations of X are useless? The answer is no. In the absence of any observation, the expected value of the square of the estimation error is:

$$\mathbf{E}\left[\left(\hat{\theta}-\Theta\right)^2\right] = (\Theta) = \frac{25}{6} - 4 = \frac{1}{6}$$

In the presence of the observation, the expected value of the square of the estimation error is:

$$\begin{aligned}
\mathbf{E}\left[\left(\hat{\theta}-\Theta\right)^2 \mid X = x\right] &= \mathbf{E}\left[\Theta^2 \mid X = x\right] - \hat{\theta}^2 \\
&= \begin{cases} \int_{3-x}^{x+1} \frac{0.5}{x-1}\theta^2 d\theta - 4 & \text{if} \quad 1 < x \le 2 \\ \int_{x-1}^{5-x} \frac{0.5}{3-x}\theta^2 d\theta - 4 & \text{if} \quad 2 \le x < 3 \end{cases} \\
&= \begin{cases} \frac{1}{3}(x-1)^2 & \text{if} \quad 1 < x \le 2 \\ \frac{1}{3}(x-3)^2 & \text{if} \quad 2 \le x < 3 \end{cases}
\end{aligned}$$

The function is plotted below.

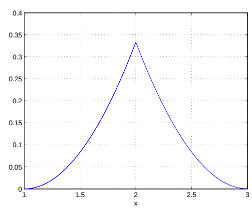

Figure 2: Plot of the Conditional Mean Squared Estimation Error as a function of x

For instance, whether the observed value of X is closer to 1 or to 2, our estimate of Θ is $\hat{\theta} = 2$. However, in the latter case ($x \approx 2$), we know that we would be potentially making a larger error than the former case ($x \approx 1$).