Recitation 22 - Solutions
December 2, 2008

1. (a) The estimator $\hat{S}_n^2$ is unbiased if the expected value of the estimator is equal to the value of the parameter being estimated. Therefore we want, $E_\lambda[\hat{S}_n^2] = v$.

$$\begin{aligned} E_\lambda[\hat{S}_n^2] &= E[c\sum_i^n (X_i - M_n)^2] \\ &= cE[\sum_i^n (X_i^2 - 2X_iM_n + M_n^2)] \\ &= c\left(E[\sum_i^n X_i^2] - nE[M_n^2]\right) \end{aligned}$$

We now solve for both $E[\sum_i^n X_i^2]$ and $E[M_n^2]$.

$$\begin{aligned} E[\sum_i^n X_i^2] &= \sum_i^n E[X_i^2], \\ &= n(\mathrm{var(X_i)} + \mathrm{E[X_i]^2}), \\ &= nv + n\lambda^2. \end{aligned}$$

$$\begin{aligned} E[nM_n^2] &= nE[(\frac{1}{n}\sum_i^n X_i)^2], \\ &= \frac{1}{n}E[(\sum_i^n X_i)^2], \\ &= \frac{1}{n}\left((n^2-n)\lambda^2 + \sum_i^n E[X_i^2]\right), \\ &= \frac{1}{n}\left((n^2-n)\lambda^2 + \sum_i^n \mathrm{var(X_i)} + \mathrm{E[X_i]^2}\right), \\ &= nE[\lambda^2 + \frac{v}{n}] \end{aligned}$$

Therefore we have

$$\begin{aligned} c &= \frac{v}{E[\sum_i^n X_i^2] - nE[M_n^2]}, \\ &= \frac{v}{v(n-1)}, \\ &= \frac{1}{n-1}. \end{aligned}$$

(b) Yes, she has enough data to calculate the ML estimate for the parameter of the poisson model. Since the mean λ is also equal to the variance v in a poisson model. The ML estimate for the mean is also the ML estimate for the variance. We now show that the ML estimator $\hat{\Lambda}_n = M_n$ thus we can draw the conclusion that the ML estimate for the variance is the value given by $m_n = 7.1$.

The PMF of X_i is

$$p_{X_i}(x) = e^{-\lambda}\frac{\lambda^x}{x!}, \qquad x = 0, 1, \ldots .$$

The log-likelihood function is

$$\log p_X(x_1, \ldots, x_n; \lambda) = \sum_{i=1}^{n} \log p_{X_i}(x_i; \lambda) = -n\lambda + \sum_{i=1}^{n} x_i \log \lambda - \sum_{i=1}^{n} \log(x_i!),$$

and to maximize it, we set its derivative to 0. We obtain

$$0 = -n + \frac{1}{\lambda}\sum_{i=1}^{n} x_i,$$

which yields the estimator

$$\hat{\lambda}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Note that this estimator is unbiased, since $\mathbf{E}[X_i] = \lambda$, so that

$$\mathbf{E}[\hat{\Lambda}_n] = \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}[X_i] = \lambda.$$

It is also consistent, because $\hat{\Lambda}_n$ converges to λ in probability, by the weak law of large numbers.

2. (a)

$$E[\hat{\Theta}_n] = \frac{1}{n}\sum_{i}^{n} E[X_i] = \theta,$$

$$\mathrm{var}(\hat{\Theta}_\mathrm{n}) = \frac{\mathrm{var}(\mathrm{X_i})}{\mathrm{n}} = \frac{\sigma^2}{\mathrm{n}}.$$

$\hat{\Theta}_n$ is gaussian because it is the sum of independent Gaussian (normal) random variables.

(b) The probability distribution of the random variable T_n under the assumption $\hat{S}_n^2 = \sigma^2$ is that of the standard normal random variable.

The event that θ lies in the confidence interval

$$[\hat{\Theta}_n - z\frac{\hat{S}_n}{\sqrt{n}}, \hat{\Theta}_n + z\frac{\hat{S}_n}{\sqrt{n}}]$$

can be written as the event

$$[-z \leq T_n \leq z].$$

Since we are interested in the 95 % confidence interval we want to find z such that $P([-z \leq T_n \leq z]) \geq 0.95$. Using the CDF of the standard normal, we have $P([-z \leq T_n \leq z]) = \Phi(z) - \Phi(-z) = \Phi(z) - 1 + \Phi(z) = 0.95$ from which we obtain $\Phi(z) = 0.975$. The value of z that attains this value is 1.96.

The confidence interval when $n = 4$ is given by,

$$[\hat{\Theta}_n - 0.98\hat{S}_n, \hat{\Theta}_n + 0.98\hat{S}_n],$$

and when $n = 16$ it is given by,

$$[\hat{\Theta}_n - 0.49\hat{S}_n, \hat{\Theta}_n + 0.49\hat{S}_n].$$

3. (a) The PDF of X_i is

$$f_{X_i}(x_i) = \begin{cases} 1, & \text{if } \theta \leq x_i \leq \theta + 1, \\ 0, & \text{otherwise.} \end{cases}$$

The likelihood function is

$$\begin{aligned} f_X(x_1, \ldots, x_n; \theta) &= f_{X_1}(x_1;\theta) \cdots f_{X_n}(x_n;\theta) \\ &= \begin{cases} 1 & \text{if } \theta \leq \min_{i=1,\ldots,n} x_i \leq \max_{i=1,\ldots,n} x_i \leq \theta + 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Any value in the feasible interval

$$[\max_{i=1,\ldots,n} X_i - 1, \min_{i=1,\ldots,n} X_i]$$

maximizes the likelihood function and is therefore a ML estimator.

(b) $\min_{i=1,\ldots,n} X_i$ converges in probability to θ, while $\max_{i=1,\ldots,n} X_i - 1$ converges in probability to θ too. We will show the first result. Let $L_n = \min_{i=1,\ldots,n} X_i$.

We intuitively expect that the series converges to θ. Note that L_n cannot increase as n increases, therefore the series is monotonic. Moreover, it is lower bounded by θ, therefore a limit exists. We therefore make a guess that it converges to θ.

Indeed for $\epsilon > 0$, we have using the independence of the X_i's,

$$\begin{aligned} \mathbf{P}(|L_n - \theta| \geq \epsilon) &= \mathbf{P}(X_1 \geq \epsilon, \cdots, X_n \geq \epsilon) \\ &= \mathbf{P}(X_1 \geq \epsilon) \cdots \mathbf{P}(X_n \geq \epsilon) \\ &= (1-\epsilon)^n. \end{aligned}$$

Since

$$\lim_{n\to\infty} \mathbf{P}(|L_n - \theta| \geq \epsilon) = \lim_{n\to\infty} (1-\epsilon)^n = 0$$

for every $\epsilon > 0$ it follows that L_n converges to θ in probability. A similar argument can be applied to show $\max_{i=1,\ldots,n} X_i$ converges in probability to $\theta + 1$.

(c) Any choice of estimator within the above interval is consistent. The reason is that $\min_{i=1,\ldots,n} X_i$ converges in probability to θ, while $\max_{i=1,\ldots,n} X_i$ converges in probability to $\theta + 1$. Thus, both endpoints of the above interval converge to θ.