Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2008)

Recitation 24 - Solutions
December 9, 2008

1. (a) We construct the LRT rule for accepting H_1 (rejecting H_0):

$$\frac{f_X(x;H_1)}{f_X(x;H_0)} \geq \xi$$
$$\frac{2x}{1} \geq \xi$$
$$x \geq \frac{\xi}{2}$$

The rejection region is therefore

$$R = \begin{cases} [\frac{\xi}{2}, 1] & \text{if } 0 \leq \xi \leq 2 \\ [0,1] & \text{if } \xi < 0 \\ \text{empty} & \text{otherwise} \end{cases}$$

(b)

$$\alpha(\xi) = \mathbf{P}(X \in R; H_0) = \int_R f_X(x;H_0)dx = \int_R 1 \cdot dx = \begin{cases} 1 - \frac{\xi}{2} & \text{if } 0 \leq \xi \leq 2 \\ 1 & \text{if } \xi < 0 \\ 0 & \text{otherwise} \end{cases}$$

(c)

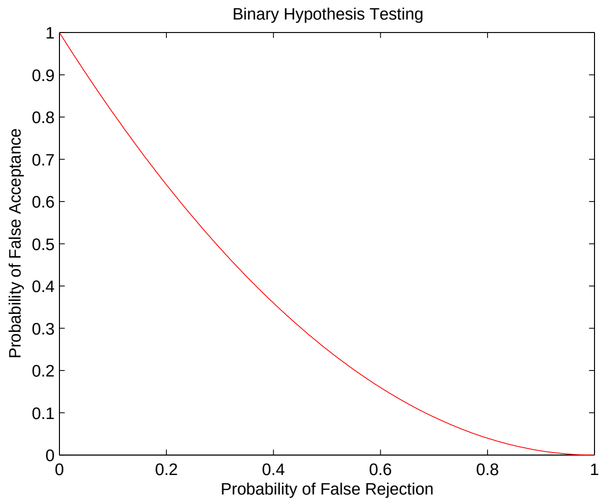
$$\beta(\xi) = \mathbf{P}(X \in R^c; H_1) = \int_{R^c} f_X(x;H_1)dx = \int_{R^c} 2x dx = \begin{cases} \frac{\xi^2}{4} & \text{if } 0 \leq \xi \leq 2 \\ 0 & \text{if } \xi < 0 \\ 1 & \text{otherwise} \end{cases}$$

(d) Clearly, the only interesting region is $0 \leq \xi \leq 2$. We can express ξ as a function of α and substitute it into the expression for β:

$$\beta = \frac{\xi^2}{4} = (1-\alpha)^2, \quad 0 \leq \alpha \leq 1.$$

The requested points on the plot:

ξ	0	2/3	1	4/3	2
α	1	2/3	1/2	1/3	0
β	0	1/9	1/4	4/9	1



(e) If we must satisfy $\alpha \leq 1/4$, it implies $\xi \geq 3/2$. The smallest value of probability of false acceptance β is achieved for $\xi = 3/2$. This corresponds to the rejection region $[3/4, 1]$ and $\beta = 9/16$.

(f) If we must satisfy $\beta \leq 1/4$, it implies $\xi \leq 1$. The smallest value of probability of false rejection α is achieved for $\xi = 1$. This corresponds to the rejection region $[1/2, 1]$ and $\alpha = 1/2$.

(g) We now construct the MAP rule for accepting H_1 (rejecting H_0):

$$\begin{aligned} \frac{p f_{X|H_1}(x)}{(1-p) f_{X|H_0}(x)} &\geq 1 \\ \frac{2x}{1} &\geq \frac{1-p}{p} \\ x &\geq \frac{1-p}{2p} \end{aligned}$$

We can see that the Bayesian hypothesis testing also reduces to the LRT. The choice of the prior probability p corresponds to a particular choice of the critical value for the LRT rule,

$$\xi = \frac{1-p}{p},$$

or, conversely,

$$\begin{aligned} \frac{1-p}{p} &= \xi \\ 1-p &= p\xi \\ p &= \frac{1}{1+\xi} \end{aligned}$$

Clearly, this only holds for $\xi \in [0, 2]$ and $p \in [1/3, 1]$. For $\xi < 0$, no valid values of p exist, and the rejection region is the entire interval $[0, 1]$ (we always decide H_1 is true). For $\xi > 2$, or $p \in [0, 1/3)$, the rejection region is empty (we always decide H_0 is true).

(h) p-**value** is the value of α for which X would be exactly at the threshold between rejection and non-rejection. Therefore we set $\xi/2 = 0.9$ and find $\alpha(\xi) = 0.1$

(a) (See solutions on pg. 515-516) Let H_0 and H_1 be the hypotheses corresponding to λ_0 and λ_1, respectively. Let X be the number of calls received on the given day. We have

$$p_X(x; H_0) = e^{-\lambda_0}\frac{\lambda_0^x}{x!}, \qquad p_X(x; H_1) = e^{-\lambda_1}\frac{\lambda_1^x}{x!}.$$

The likelihood ratio is

$$L(x) = \frac{p_X(x; H_1)}{p_X(x; H_0)} = e^{\lambda_0 - \lambda_1}\left(\frac{\lambda_1}{\lambda_0}\right)^x.$$

The rejection region is of the form

$$R = \{x \mid L(x) > \xi\},$$

or by taking logarithms,

$$R = \{k \mid \log L(x) > \log \xi\} = \{x \mid \lambda_0 - \lambda_1 + x(\log \lambda_1 - \log \lambda_0) > \log \xi\}.$$

Assuming $\lambda_1 > \lambda_0$, we have

$$R = \{x \mid x > \gamma\},$$

where

$$\gamma = \frac{\lambda_0 - \lambda_1 + \log \xi}{\log \lambda_1 - \log \lambda_0}.$$

To determine the value of γ for a probability of false rejection equal to α, we must have

$$P(x > \gamma; H_0) = 1 - F_X(\gamma; H_0) \le \alpha,$$

Where $F_X(\,\cdot\,; H_0)$ is the CDF of the Poisson with parameter λ_0.

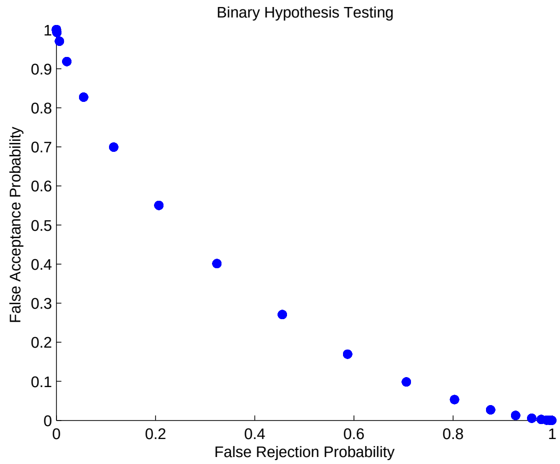
(b)

$$P(x > \gamma; H_0) = 1 - F_X(\gamma; H_0) \le \alpha$$

The smallest value of γ for which the above holds is 9. Therefore,

$$\beta = \mathbf{P}(x \le \gamma; H_1) = F_X(\gamma; H_1) = 0.5874.$$

The following diagram shows the plot of P_M vs P_{FA}. You can confirm the above answer from the plot.


Binary Hypothesis Testing
False Acceptance Probability
False Rejection Probability
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1