Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2008)

Tutorial 1 Solutions
September 11/12, 2007

1. (a) In order to wind up in the same place after two steps, the tightrope walker can either step forwards, then backwards, or vice versa. Therefore the required probability is:

$$2 \cdot p \cdot (1-p).$$

(b) The probability that after three steps he will be one step ahead of his starting point is the probability that out of 3 steps in total, 2 of them are forwards, and one is backwards. This equals:

$$\binom{3}{1} \cdot p^2 \cdot (1-p).$$

(c) Given that out of his three steps only one is backwards, the sample space for the experiment is:

$$\{(F,F,B);(F,B,F);(B,F,F)\}$$

where F denotes a step forwards, and B a step backwards. Each of these sample points is equally likely, therefore the probability that his first step is a step forward is $\frac{2}{3}$.

2. Initially, there are 10 forks and no knives in the left drawer, which we denote by L(10F,0K). Similarly, the right drawer has R(0F,10K). After the roommate takes two forks out of the left drawer and places them in the right drawer, the composition of the two drawers becomes L(8F,0K) and R(2F,10K). For the roommate's second action from the right drawer, the probability of pulling a fork is $2/12 = 1/6$, and the probability of pulling a knife is $10/12 = 5/6$.

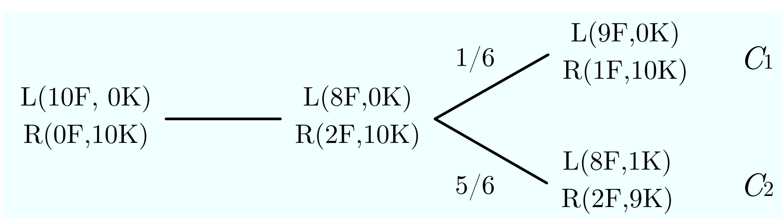

Figure 1: Tree diagram for problem 2

From the initial configuration, the system may have evolved in two possible ways (Please see Fig. 1.) We call these cases C_1 and C_2. Case C_1 refers to a fork being transferred from the right drawer to the left, leaving the two drawers as L(9F,0K) and R(1F,10K). C_2 refers to a knife transferred to the left, resulting in L(8F,1K) and (2F,9K).

From the above argument, $\mathbf{P}(C_1) = 1/6$, and $\mathbf{P}(C_2) = 5/6$.

From Bayes's Rule,

$$\mathbf{P}(\text{left chosen}|\text{knife pulled}) = \frac{\mathbf{P}(\text{knife pulled}|\text{left chosen})\mathbf{P}(\text{left chosen})}{\mathbf{P}(\text{knife pulled})}$$

Using the Total Probability Theorem,

$$\begin{aligned}\mathbf{P}(\text{knife}|\text{left chosen}) &= \mathbf{P}(\text{knife}|C_1, \text{left chosen})\mathbf{P}(C_1) + \mathbf{P}(\text{knife}|C_2, \text{left chosen})\mathbf{P}(C_2) \\ &= 0 \cdot \frac{1}{6} + \frac{1}{9} \cdot \frac{5}{6} = \frac{5}{54}\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbf{P}(\text{knife}|\text{right chosen}) &= \mathbf{P}(\text{knife}|C_1, \text{right chosen})\mathbf{P}(C_1) + \mathbf{P}(\text{knife}|C_2, \text{right chosen})\mathbf{P}(C_2) \\ &= \frac{10}{11} \cdot \frac{1}{6} + \frac{9}{11} \cdot \frac{5}{6} = \frac{5}{6}\end{aligned}$$

Using Bayes's Theorem,

$$\begin{aligned}\mathbf{P}(\text{left chosen}|\text{knife}) &= \frac{\mathbf{P}(\text{knife}|\text{left chosen})\mathbf{P}(\text{left chosen})}{\mathbf{P}(\text{knife})} \\ &= \frac{5/54 \cdot 1/2}{5/54 \cdot 1/2 + 5/6 \cdot 1/2} \\ &= \frac{1}{10}\end{aligned}$$

3. The probability that persons 1 and 2 both roll a particular face is $1/n^2$. Therefore,

$$P(A_{12}) = P(A_{13}) = P(A_{23}) = \frac{n}{n^2} = \frac{1}{n}.$$

Similarly, we also have

$$P(A_{12} \cap A_{13}) = P(\text{all players roll the same face}) = \frac{n}{n^3} = \frac{1}{n^2},$$

so

$$P(A_{12} \cap A_{13}) = P(A_{12}) \cdot P(A_{13}).$$

Hence A_{12} and A_{13} are independent, and the same is true of any other pair from the events A_{12}, A_{13}, and A_{23}. However, A_{12}, A_{13}, and A_{23} are not independent. In particular, if A_{12} and A_{13} occur, then A_{23} also occurs.

4. **(Galton's Paradox)** The explanation is incorrect; the correct probability is 2/8. There are 8 possibilities when 3 coins are flipped: HHH, HHT, HTH, HTT, THH, THT, TTH, and TTT. Obviously, only the first and last have coins that are "all alike."

   The argument given is flawed: obviously, at least two must be alike, but the statement doesn't talk about *which two*. If the first two are alike, then *conditioned on that information*, the probability that they're all alike is 1/2. But, the first two coins could just as well be different, in which case all 3 are not alike.